Doctoral School of Exact and Natural Sciences – Physical Sciences Examination

In the solutions, present the reasoning leading to your results. Write down the final results of the calculations with accuracy of 3 or 2 significant digits, after appropriate rounding, for example $1.23456 \cdot 10^{-19} \approx 1.23 \cdot 10^{-19}$ or $1.2 \cdot 10^{-19}$.

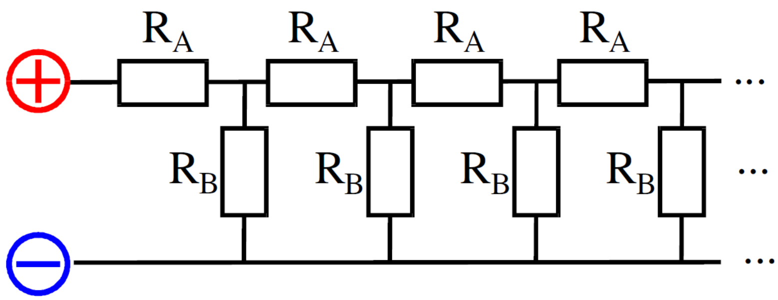
Values of selected constants

speed of light in vacuum	$c \approx 3.00 \cdot 10^{8}$ m/s
elementary charge	$e \approx 1,60 \cdot 10^{-19}$ C
Coulomb constant	$k_e \approx 8.99 \cdot 10^{9}$ N·m^2/C^2
Planck constant	$h \approx 6.63 \cdot 10^{-34}$ Js $\approx 4.14 \cdot 10^{-15}$ eVs
reduced Planck constant	$\hbar = \frac{h}{2\pi} \approx 1.05 \cdot 10^{-34}$ Js $\approx 6.58 \cdot 10^{-16}$ eVs
gravitational constant	$G \approx 6.67 \cdot 10^{-11}$ Nm^2/kg^2
Avogadro constant	$N_A \approx 6.02 \cdot 10^{23}$ mol^{-1}
gas constant	$R \approx 8.31$ J / (mol·K)
Boltzmann constant	$k_B \approx 1.38 \cdot 10^{-23}$ J/K$\approx 8.62 \cdot 10^{-5}$ eV/K
Rydberg constant	$R_\infty \approx 1.10 \cdot 10^{7}$ m^{-1}
Rydberg unit of energy	$Ry \approx 13.6$ eV
electron mass	$m_e \approx 9.11 \cdot 10^{-31}$ kg ≈ 511 keV/c^2
proton mass	$m_p \approx 1.67 \cdot 10^{-27}$ kg ≈ 938 MeV/c^2
unified atomic mass unit	$u \approx 931$ MeV/c^2

**Problems 1—7 are easier. Submit solutions of only four of them.
For each of them you can get up to 6 points.
Problems 8-12 are more difficult. Submit solutions of only two of them.
For each of these solutions you can get up to 8 points.**

Easier problems

Problem 1. Resistors
The network of resistors shown in the figure consists of an infinite number of resistors R_A and R_B. Find the total resistance R of this network.



Problem 1. Infinite circuit of resistors.

Problem 2. Earth falling onto the Sun
Imagine that the orbital movement of the Earth was stopped and the Earth started to fall freely onto the Sun. Find the total time T of free fall before the Earth hits the Sun. One year is $T_0 \approx 3.156 \cdot 10^7$ s.

Problem 3. Ratio of momenta
Let p_f denote the momentum of photon of energy $E = 1.17$ eV and p_e the momentum of electron with kinetic energy also equal $E = 1.17$ eV. Find the ratio p_f/p_e. The rest mass of electron $m_e = 511$ keV$/c^2$.

Problem 4. Polarizers
The passage of a parallel beam of unpolarized light through linear polarizers was studied. The beam intensity was I_0 (units: W/m^2). The polarizers were inserted into the beam in such a way that they were perpendicular to the beam. All polarizers were identical.

(a) First, only one polarizer was inserted into the beam, The light intensity behind this polarizer was equal to $(46 \pm 1)\%$ of the intensity of the initial beam. Name at least two phenomena that may have accounted for a deviation of this result from 50%.

(b) A second polarizer was inserted after the first polarizer. The planes of polarization of both polarizers were parallel. The ratio of the intensity of light passing through the second polarizer to the intensity of the light falling on it was equal to $(94 \pm 2)\%$. Is the result of this measurement consistent with the result in (a)? Justify your answer by performing the necessary basic calculations.

(c) Write down the formula for determining the intensity of light after it has passed through the following set of N polarizers. The unpolarized beam falls on the first polarizer. The plane of polarization of each subsequent polarizer is at angle α to the plane of polarization of the previous polarizer. Include the parameter describing the effect observed in (a) and (b).

Problem 5. Activity of a sample
What is the activity of a sample of ^{210}Po (emitter of α particles) of mass $m = 1 \mu$g. The half-life of ^{210}Po is approximately 138 days.

Hint: Activity denotes the number of radioactive decays per second.

Problem 6. Eigenvalues of the energy
Some system is governed by the following Hamiltonian:

$$\hat{H} = \frac{1}{2}\hbar\omega(\hat{a} + i\hat{b})(\hat{a}^\dagger - i\hat{b}^\dagger),$$

where ω is a frequency, while, $\hat{a}$ i $\hat{b}$ are annihilation operators of two independent degrees of freedom. Find the energy eigenvalues of this system, if the following commutation rules are fulfilled:

$$[\hat{a}, \hat{a}^\dagger] = 1 = [\hat{b}, \hat{b}^\dagger], \quad [\hat{a}, \hat{b}^\dagger] = 0 = [\hat{a}, \hat{b}].$$

Additionally, the n-th eigenstate of the excitation number operator, $\hat{a}^\dagger\hat{a}$, of system a fulfills the relations:

$$\hat{a}^\dagger\hat{a}|n_a, n_b\rangle = n_a|n_a, n_b\rangle, \quad \hat{a}|n_a, n_b\rangle = \sqrt{n_a}|n_a - 1, n_b\rangle, \quad \hat{a}^\dagger|n_a, n_b\rangle = \sqrt{n_a + 1}|n_a + 1, n_b\rangle$$

and analogous relations hold for system b:

$$\hat{b}^\dagger\hat{b}|n_a, n_b\rangle = n_b|n_a, n_b\rangle, \quad \hat{b}|n_a, n_b\rangle = \sqrt{n_b}|n_a, n_b - 1\rangle, \quad \hat{b}^\dagger|n_a, n_b\rangle = \sqrt{n_b + 1}|n_a, n_b + 1\rangle$$

Problem 7. Poisson distribution
It has been observed that during a shift, a doctor is called on average 3 times. What is the probability that the shift will pass without the doctor being called? Assume that the number of calls during one shift follows the Poisson distribution. Hint: Poisson distribution is a discrete probability distribution with parameter λ, described by the formula:

$$P_\lambda(k) = \frac{\lambda^k}{k!}e^{-\lambda}.$$

More difficult problems

Problem 8. Cyclist in a cross-wind
A cyclist is riding with the velocity $v = 8$ m/s along a straight, horizontal road. The forest on both sides of the road shields the cyclist from the wind. Outside the forest, the wind is blowing with the velocity $u = 6$ m/s, perpendicular to the road. By what factor must the cyclist increase his power output to maintain the constant speed after leaving the shelter of the forest?
Hint: The air resistance force is proportional to the square of the velocity of the moving object relative to the air. Assume that the value of this force does not depend on the direction of the object's motion relative to the air.

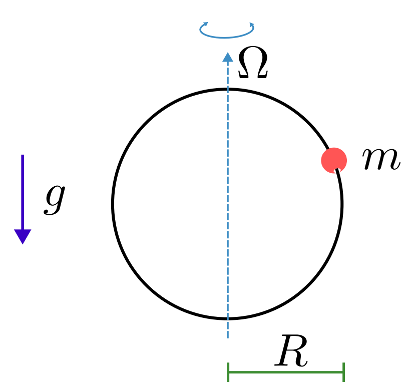
Problem 9. Ice from supercooled water
Water devoid of impurities and left without mechanical shocks can be supercooled to the temperature $T_H \approx 225$ K (–48 ℃) and remain in its liquid state (known as the supercooled state). At temperature $T \leq T_H$ ice crystals form and grow rapidly in the liquid water. A similar phenomenon occurs when supercooled water at $T > T_H$ is disturbed, for example by removing it from a freezer. What mass, m_l, of ice will form in a bottle containing $m = 0.5$ kg of water supercooled in the freezer? The temperature in the freezer is $T_0 = 263.15$ K (–10 ℃). Data

for water: melting temperature, $T_m = 273.15$ K, heat of fusion, $L = 333.6$ J/g, specific heat capacity of supercooled water, $c_w = 4.2$ J/(g·K), specific heat capacity of ice, $c_l = 2.1$ J/(g·K). The formation and growth of ice crystals are so rapid that heat exchange with the surroundings (through the walls of the bottle) can be neglected.

Problem 10. Mass of the neutrino.
Neutrinos are elementary particles of very small rest mass and very weak interactions with matter, which makes them very difficult to study. On February 23, 1987 in the Kamiokande detector, a burst of neutrinos lasting for $\Delta t \approx 2$ s was recorded, and the energies of the registered neutrinos ranged from approximately 8 MeV to approximately. 40 MeV. It was assumed that they were emitted during the supernova explosion SN1987A, which occurred at a distance from the Earth of $L \approx 170\ 000$ light-years. It is assumed that all observed neutrinos were of the same type (electron neutrinos) and were emitted simultaneously. Based on this, estimate their rest mass m_ν. One year is $T_0 \approx 3.156 \cdot 10^7$ s.

Problem 11. Absorbancy
The absorption spectrum of a solution containing protein and nucleoside was measured by placing the solution in a quartz cuvette with an optical path length $l = 0.5$ cm. The absorbance at wavelengths of 280 nm and 260 nm was found to be $A_{280nm} = 0.72$ and $A_{260nm} = 0.68$ respectively, and the background absorbance is $A_{\lambda>340nm} = 0.02$.
Subsequently, the solution of this mixture was diluted twice with the same solvent, which was used to prepare the initial solution, and the absorption spectrum was measured again using the same quartz cuvette. This time, the absorbance at 280 nm and 260 nm was $A_{280nm} = 0.37$ and $A_{260nm} = 0.35$ respectively, and the background absorbance remained the same.
Determine the concentration of each of the components of the mixture present in the solution, knowing that under the measurement conditions, decimal molar absorption coefficients at 260 nm and 280 nm are as follows: $\varepsilon_{260nm}^{protein} = 12\ 000$ M^{-1}cm^{-1} for the protein and $\varepsilon_{260nm}^{nucleoside} = 14\ 000$ M^{-1}cm^{-1} for the nucleoside, and $\varepsilon_{280nm}^{protein} = 35\ 000$ M^{-1}cm^{-1} for the protein, while the nucleoside does not absorb at this wavelength. Justify your answer.

Problem 12. Rotating ring
A point-like mass m can side without any friction along a circular ring of radius R that is rotating with the angular frequency Ω around the vertical axis passing through the center of the ring (as shown in the figure below). The mass m is subjected to the external gravitational field with acceleration g.
Find the equilibrium positions for the point m and verify their stability. Next, find the frequency of small oscillations around the stable equilibrium points.



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