Warsaw Doctoral School of Mathematics and Computer Science

June 9th, 2022

Entrance Exam

On the following pages you will find 16 problems related to various areas of Mathematics and Computer Science. You are expected to choose and solve any 4 of them. Each problem is worth the same number of points.

You are free to choose any problems you wish, i.e., candidates for studies in Mathematics may choose also Computer Science problems and vice versa.

Every problem is composed of a few subproblems of comparable size, but each problem (i.e. all its subproblems) is graded as a whole.

You can attempt to solve more than 4 problems. All your solutions will be graded, but only 4 best-graded solutions will contribute to your general grade.

All your answers should be appropriately justified.

Every problem should be solved on a separate sheet of paper, signed with your full name and marked with the problem number.

Time: 3 hours.

Good luck!

1. **Linear Algebra**

   Let V be a finite dimensional vector space over real numbers, let $\langle \cdot, \cdot \rangle$ be a scalar product on V. Denote $\mathrm{End}(V)$ the space of linear endomorphisms of V.

   a) Prove that the map $S : \mathrm{End}(V) \to \mathrm{End}(V)$ given by the formula $S(T) = T^*$ is linear and satisfies $S(TW) = S(W)\,S(T)$.

   b) Prove that the formula $(T, W) := Tr(T^* W)$ defines positively defined scalar product on $\mathrm{End}(V)$, where Tr is a trace of endomorphism.

   c) Is the map S a self-adjoint map?

   d) Describe eigenvalues and eigenvectors of the map S.

   e) Assume that $A \in \mathrm{End}(V)$ self-adjoint, $T \in \mathrm{End}(V)$ has the following property: every eigenvector of A is simultaneously an eigenvector of T. Whether necessarily $AT = TA$? Is the assumption that A is self-adjoint necessary?

   f) Assume that T satisfies $T^2 = T$. Prove that T is an orthogonal projection if and only if T is self-adjoint.

2. **Algebra**

   Consider the ring $R = \mathbb{Z}_{18}[x]$.

   a) How many nilpotent elements of the form $x^{10} + bx^2 + c$, where $a, b, c \in \mathbb{Z}_{18}$ are in R?

   b) How many invertible elements of the form $sx^7 + u$, where $s, u \in \mathbb{Z}_{18}$ are in R? Give an example $s, u \in \mathbb{Z}_{18}$, $s \neq 0$ such that $sx^7 + u$ is invertible in R and calculate this inverse element.

   c) Prove that the ideal $I = (x^2+1)$ is not prime in R. Give any maximal ideal containing I.

3. **Analysis**

   Consider the following system of equations

$$\begin{cases} u \ln x + y e^{u+w} = e, \\ e^{2w} + w \ln(xyz) = e^{-2}. \end{cases} \qquad (\star)$$

   Let

$$B_r^3 = \left\{ (x, y, z) \in \mathbb{R}^3 : (x-1)^2 + (y-1)^2 + (z-1)^2 < r^2 \right\}$$

   a) Prove that the system of equations $(\star)$ in a neighbourhood of $p = (x_0, y_0, z_0, u_0, w_0) = (1, 1, 1, 2, -1)$ defines uniquely coordinates u, w as a functions of (x, y, z). Prove that there exists $R > 0$ such that the functions $u(x, y, z)$ and $w(x, y, z)$ defined by $(\star)$ are of class C^2 on ball B_R^3.

   b) Calculate deriatives $\frac{\partial u}{\partial y}(1, 1, 1)$ and $\frac{\partial w}{\partial z}(1, 1, 1)$.

   c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function such that

$$\mathrm{grad} f(1, -2) = [e, -e].$$

   Let $h(x, y, z) = f\big(u(x, y, z), w(x, y, z)\big)$.

   - Whether the function h has a local extremum at the point $(1, 1, 1)$?
   - Whether the function h attains its infimum and supremum on $\bar{B}^3_{R/2}$, where $\bar{A}$ is a closure of A.

d) Let $\omega \in \Omega^2(B^3_R)$ be a differential form give by the formula

$$\omega(x,y,z) = \frac{\partial u(x,y,z)}{\partial x}\, \mathrm{d}x \wedge \mathrm{d}y - \frac{\partial u(x,y,z)}{\partial z}\, \mathrm{d}y \wedge \mathrm{d}z.$$

Let $r < R$. Compute the integral along the semi-sphere:

$$\int_{S_r^+} \omega,$$

where $S_r^+ = \{(x,y,z) \in \mathbb{R}^3 : (x-1)^2 + (y-1)^2 + (z-1)^2 = r^2, \quad y \geq 1\}$. The orientation of S_r^+ is given by continuous normal vector field $\overline{n}$ such that $\overline{n}(1, 1+r, 1) = (0,1,0)$.

4. **Topology**

A metric space (X,d) is called ultrametric, if, for $x, y, z \in X$, the metric d satisfies the following stronger form of the triangle inequality:

$$d(x,z) \leq \max(d(x,y), d(y,z)).$$

a) Prove that, for a given prime number p, the p-adic metric d defined on the set of integers $\mathbb{Z}$ by

$$d(a,b) = \min\{2^{-i} : p^i \text{ divides } |a-b|\},$$

for $a \neq b \in \mathbb{Z}$, is an ultrametric. Are there any limit points in $(\mathbb{Z}, d)$?

b) Show that, if B, B' are balls in an ultrametric space X with the same radius $r > 0$, then $B = B'$ or $B \cap B' = \emptyset$. Prove that the distance set $O = \{d(x,y) : x, y \in X\}$ of a compact ultrametric space X is either finite or has the form of a sequence converging to 0.

c) Using the first part of b), charactrize connected subspaces of an ultrametric space.

d) Show that a complete and countable metric space (not necessarily ultrametric) has an isolated point.

5. **Probability**

An urn contains: 1 white ball, 1 black ball, 1 green ball. We draw balls from the urn consecutively with replacement until the black ball is selected.

a) What is the probability that the number of draws will exceed 10?

b) What is the probability that the numbers of white and green balls draws are the same?

c) Calculate the expected value of of the number of white ball draws.

d) Calculate the expected value of of the number of draws, provided the green ball was not drawn.

e) Calculate the probability that the number of draws is 7, provided the numbers of white and green balls draws are the same

6. **Functional Analysis**

Consider the following integral transform

$$Tf(x) = \int_{-\infty}^{+\infty} e^{-t^4 x} f(t)\, dt.$$

a) Compute $Tf(x)$ fot $f(t) = t$ and $x > 0$.

b) Assume $f \in L_p(\mathbb{R})$ for $1 \le p \le \infty$. Prove that $Tf(x)$ is well defined for $x > 0$.

c) Prove that if $f \in L^\infty(\mathbb{R})$ then

$$|Tf(x)| \le C x^{-\frac{1}{4}},$$

where C depends only on f.

d) Prove that if $f \in L^p(\mathbb{R})$ for $1 < p < \infty$, then

$$|Tf(x)| \le C x^{-\frac{p-1}{4p}},$$

where $C = C(f, p)$.

e) Assume that $f \in L_p(\mathbb{R})$ for $1 \le p \le \infty$.Prove that $Tf \in C^1((0, +\infty))$. Find the formula for $(Tf)'(x)$.

7. **Differential Equations**

a) Find all solutions of the following Cauchy problem

$$\begin{cases} x'(t) = x^{3/4} & \text{for} \quad t > 0, \\ x(0) = 0 \end{cases}$$

b) Find the general solution of the following equation

$$x'' - 2x' + x = t, \quad x(0) = 1.$$

Does there exist a solution $x(t)$ such that $x(1) = 1$?

c) Consider the following problem

$$\begin{cases} x'(t) = ax^6 - bt & \text{dla} \quad t > 0, \\ x(0) = x_0 > 0, \end{cases}$$

where a and b are positive constants. Prove that there exists $x^* > 0$ such that if $x_0 < x^*$ then the solution $x(t)$ is well defined for all $t > 0$.

8. **Analytic Functions**

a) Prove that $h(z) = \frac{1}{z-1}$ is the unique holomorphic function on $\mathbb{C} \setminus \{1\}$ that satisfies the following conditions:

- the map $h\colon \mathbb{C} \setminus \{1\} \to \mathbb{C} \setminus \{0\}$ is a bijection,
- $h(0) = -1$.

b) Let $n \in \mathbb{N}$, $F(z) = h(z^n)$, $C_R = \{z \in \mathbb{C} : \ |z| = R\}$, $I_R = \int_{C_R} F(z)dz$ (on the circle C_R we assume counterclockwise orientation).
Compute I_R, where $R < 1$.

c) Compute I_R for $R > 1$ and $n = 1$.

d) Compute I_R for $R > 1$ and $n \ge 2$.

9. **Discrete Mathematics**

There is an undirected graph where the set of vertices is given as $\{(i,j)\colon i,j \in \{0,1,\ldots,n\}\}$. There are $2n(n+1)$ endges connecting neighbouring points: that is for each $i \in \{0,1,\ldots,n\}$ and $j \in \{0,1,\ldots,n-1\}$ we have an edge between (i,j) and $(i,j+1)$, and an edge between (j,i) and $(j+1,i)$.

(a) What is the number of shortest paths connecting $(0,0)$ and (n,n)?

(b) Let us fix three edges: e_1 which connects (i_1,j_1) and (i_1+1,j_1), (2) e_2 connecting (i_2,j_2) with (i_2+1,j_2), and (3) e_3 connecting (i_3,j_3) with (i_3+1,j_3). We have $0 \le i_1 < i_2 < i_3 < n$ and $0 \le j_1 < j_2 < j_3 \le n$. What is the number of shortest paths connecting $(0,0)$ and (n,n), that contain at least one of the edges e_1, e_2 and e_3? It is sufficient to give an answer in the form of a sum.

(c) Consider the following subset of vertices: $S = \{(i,j)\colon i,j \in \{n_\ell, n_\ell+1,\ldots,n_u\}\}$, where $0 < n_\ell < n_u < n$. What is the number of shortest paths connecting $(0,0)$ and (n,n) that go through at least one vertex from S. It is sufficient to give an answer in the form of a sum.

(d) Let us fix a strictly increasing sequence of $m+1$ natural numbers $k_0,k_1,\ldots,k_m$, where for each $i \in \{0,1,\ldots,m\}$ we have $k_i \in \{0,1,\ldots,n\}$. Prove the following inequality:

$$\sum_{i=0}^{m} \binom{n-k_i+i}{i}\binom{m+k_i-i}{k_i} \le \binom{n+m}{n}.$$

What is the condition for the equality to hold?

10. **Complexity theory**

The problem VERTEXCOVER is defined as follows. We are given an undirected graph $G = (V,E)$ and an integer k. We will say that a subset of vertices $S \subseteq V$ covers a vertex $v \in V$ if $v \in S$ or if there is an edge between v and some vertex from S. We ask if there exists a subset of vertices $S \subseteq V$ of size k ($|S| = k$) that covers all the vertices in G. In the optimisation variant of the problem we ask for the subset of vertices $S \subseteq V$ of size k that covers the greatest number of vertices in G. The problem VERTEXCOVER is NP-hard.

(a) Prove that VERTEXCOVER remains NP-hard even if we restrict the input to the graphs for which the degree of each vertex is greater than or equal to 2022.

(b) Assume that the graphs have the following property: If we enumerate the degrees of all the vertices in the graph, then we will get at least $|V| - 2022$ different numbers. Does VERTEXCOVER remain NP-hard for such graphs?

(c) Consider the following local-search algorithm for VERTEXCOVER. We start with an arbitrary subset $S \subseteq V$, $|S| = k$. Next, in the loop we perform the following operation. If there exist two vertices $v \in S$ and $v' \notin S$ such that set $(S \setminus \{v\}) \cup \{v'\}$ covers more vertices than S then we do the swap (we remove v from S and add v' to S; $S := (S \setminus \{v\}) \cup \{v'\}$). If there exist no such two vertices, then we stop and return S. Prove that this is a 2-approximation algorithm for VERTEXCOVER.

(d) Consider the parameter p defined as follows: 2022 vertices have degrees greater than $|V| - p$, and the remaining vertices have degrees lower than p. Is VERTEXCOVER FPT (fixed-parameter tractable) for the parameter p? Justify your answer.

11. **Algorithms and data structures**

Given a tree T with edge weights $w(e) > 0$ for each $e \in E(T)$. Let L be the set of all leaves in T. For $X \subseteq L$ let $T|X$ be the minimal connected subgraph of T having all leaves from X. The weight of $T|X$ is defined as $w(X) = \sum_{e \in E(T|X)} w(e)$. For $k > 0$ let $p_k(T) = \max\{w(X) \colon X \subseteq L \text{ and } |X| = k\}$.

(a) Show that, for any sets X, $X' \subseteq L$ such that $|X| > |X'| \geq 2$, there is $x \in X \setminus X'$, such that,

$$w(X \setminus \{x\}) + w(X' \cup \{x\}) \geq w(X) + w(X'). \tag{1}$$

(b) Using (1), prove the following statement: if $w(X) = p_k(T)$ for $2 \leq |X| = k < n$, then there is x such that $w(X \cup \{x\}) = p_{k+1}(T)$.

(c) Based on the previous observation propose an $O(n^2)$-time algorithm for the computation of $p_k(T)$, where n is the size of T.

(d) Propose an $O(n \log n)$-time algorithm for the computation of $p_k(T)$.

12. **Logic and Databases**

Are the following decision problems decidable?

(a) Does a given sentence of the first-order logic (over any signature) has a 2022-element model?

(b) Is a given sentence of the first-order logic over the signature consisting of a ternary relational symbol *triangle* (and equality) true in natural numbers, where *triangle*(x, y, z) holds when one can build a triangle from segments of length x, y, z?

(c) Is a given sentence of the first-order logic over the signature consisting of one binary functional symbol f (and equality) true in natural numbers, where f is interpreted as the function $(x, y) \mapsto (x + y)^2$?

(d) Is a given sentence of the first-order logic sentence over the signature $(+, \cdot, 0, 1)$ true in the field of complex numbers?

13. **Automata and formal languages**

Do there exist regular languages $L_n \subseteq \{a, b\}^*$ having the following properties? For each item, provide an example (and justify its correctness) or prove that no such languages exist.

(a) Every deterministic finite automaton recognizing L_n has at least $\Theta(2^n)$ states, but L_n can be recognized by a nondeterministic finite automaton having $O(n)$ states.

(b) Every deterministic finite automaton recognizing L_n has at least $\Theta(2^n)$ states, but there exists $K_n \subseteq (\{a, b\} \times \Sigma)^*$ (for some alphabet Σ) such that $L_n = \pi_1(K_n)$ and K_n can be recognized by a deterministic finite automaton having $O(n)$ states (the π_1 operation removes the second coordinate in each letter of each word from K_n, leaving only a or b).

(c) Every nondeterministic finite automaton recognizing L_n has at least $\Theta(2^n)$ states, but L_n^R (the set of words from L_n written backwards) can be recognized by a nondeterministic finite automaton having $O(n)$ states.

(d) Every nondeterministic finite automaton recognizing L_n has at least $\Theta(2^n)$ states, but L_n can be recognized by a nondeterministic pushdown automaton having $O(n)$ states.

14. **Concurrent and distributed programming, computer systems**

We consider the model of asynchronous communication:

- A call to `send(p,m)` queues a message `m` destined for a process `p`. The call ends immediately, without waiting for the message to be delivered.
- A call to `get()` executed by a process p returns the first queued message destined for p. Messages for p are received in the order in which they are sent. If no message for p is queued, the function waits for such a message to arrive.

Consider a server process that manages access to two resources, A, B. For each of them, it remembers whether the resource is available or busy. It also stores a list of waiting requests from other processes. This process does the following in an endless loop:

- Receive a message `m` using the function `get()`.
- If `m` is `freeA` or `freeB`, then indicate that the resource is available.
- Otherwise, add `m` to the end of the list of waiting requests. The requests are of the form (`takeA`, p) or (`takeB`, p) and say that a process p wants to access resource A or B.
- In both cases, on the list of waiting requests find the first request that can be satisfied. If it exists, then indicate that the resource is busy, remove the request from the list, and send a message `start` to the requesting process p.

(a) We run multiple copies of a process described by the following pseudocode:

```
while (true) {
  send(serv, takeA)
  send(serv, takeB)
  get()
  get()
  use_A_and_B()
  send(serv, freeA)
  send(serv, freeB)
  work_yourself()
}
```

What can go wrong (why the client code written above may not work as expected)?

(b) Is it possible to fix the above code without changing the server code? We assume that when any of the processes calls `use_A_and_B`, the server should know that both resources A and B are busy.

(c) Now suppose that the server can also handle a request of the form (`takeAB`, p), which says that a process p wants to access both resources A and B (if the server, while scanning the list of waiting requests, finds such a request and both resources are available, the server indicates that they are busy, removes the request from the list, and sends a message `start` to the requesting process p). We run multiple copies of three kinds of processes, described by the following pseudocode:

```
while (true) {
  send(serv, takeA)
  get()
  use_A()
  send(serv, freeA)
  work_yourself()
}
```

```
while (true) {
  send(serv, takeB)
  get()
  use_B()
  send(serv, freeB)
  work_yourself()
}
```

```
while (true) {
  send(serv, takeAB)
  get()
  use_A_and_B()
  send(serv, freeA)
  send(serv, freeB)
  work_yourself()
}
```

What can go wrong (why the client code written above may not work as expected)?

(d) Is it possible to fix the above code without changing the server code?

15. **Programming languages**

(a) Provide the exact output of the function `f` in C for the given input values `a=21, b=51`:

```
int f(int a, int b) {
      if (a > b)
          return f(a - b, b);
      if (b > a)
          return f(a, b - a);
      return a;
}
```

Write a variant of function `f` without using the recursive calls.

(b) What will be the result of the class `Test` in Java? What exactly is printed out? Justify your answer. Describe problems that you notice here.

```
public class Test {
    public static void main(String[] args) {
        A red = new A("Red");
        A blue = new A("Blue");
        f(red, blue);
        System.out.println(red.getColor());
        System.out.println(blue.getColor());
        ff(blue);
        System.out.println(blue.getColor());
    }
    public static void f(Object o1, Object o2) {
        Object temp = o1;
        o1 = o2;
        o2 = temp;
    }
    private static void ff(A a) {
        a.setColor("Red");
        a = new A("Green");
        a.setColor("Yellow");
    }
}
class A {
    private String color;
    public A(String color) {
        this.color = color;
    }
    public String getColor() {
        return color;
    }
    public void setColor(String color) {
        this.color = color;
    }
}
```

(c) Consider the following functions implemented in Ocaml:

```
let rec f1 g n = if n = 1 then g else fun x -> g ((f1 g (n-1)) x);;

let f2 g n = fun x ->
    let rec h k m r = if m = 0 then r else h k (m-1) (k r) in h g n x;;
```

Describe what f1 and f2 return? What is the time complexity and space complexity of functions f1 and f2? What is the difference between these two functions?

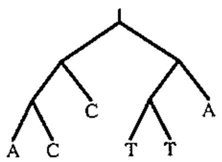
(d) Consider the following class implementing trees in Python 3:

```
class Tree:
    def __init__(self):
        self.left = None
        self.right = None
        self.data = None
```

Write a generator that traverses a given tree inorder, that is it first traverses the left subtree, then goes to the root and then, traverses the right subtree.

16. **Bioinformatics and computational biology**

(a) Under the assumption of the classical maximum parsimony compute the cost (i.e., the number of required mutations) and all optimal node labellings for the tree depicted below, where each leaf has a given sequence of the length one.



(b) Is it possible to fill all missing cells in the matrix depicted below, such that the obtained matrix is additive? Justify your answer.

			6	
	8			
6	6	6		

(c) Is it possible to fill all missing cells in the matrix from the previous question (16b), such that the obtained matrix is ultrametric? Justify your answer.

(d) Given a distance matrix M. Does UPGMA[1] method infer a tree from M, such that the distance between i and j equals $M[i, j]$, for every pair of leaves i and j? Justify your answer.

[1]Unweighted Pair Group Method with Arithmetic mean