Warsaw Doctoral School of Mathematics and Computer Science

Entrance exam, 12 July 2021

On the following pages you will find 16 problems related to various areas of Mathematics and Computer Science. You are expected to choose and solve any 4 of them. Each problem is worth the same number of points.

You are free to choose any problems you wish, i.e., candidates for studies in Mathematics may choose also Computer Science problems and *vice versa*.

Every problem is composed of a few subproblems of comparable size, but each problem (i.e. all its subproblems) is graded as a whole.

You can attempt to solve more than 4 problems. All your solutions will be graded, but only 4 best-graded solutions will contribute to your general grade.

All your answers should be appropriately justified.

Every problem should be solved on a separate sheet of paper, signed with your full name and marked with the problem number.

Good luck!

1. **Analysis**

Let us consider a function $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(x) = \sum_{n=1}^{\infty} \frac{n\, x^2}{(1+x^2)^{n+1}}\,.$$

(i) Find an explicit form of $f(x)$ without infinite summation.

(ii) Is this series uniformly convergent on $\mathbb{R}$? Is this series almost convergent on $\mathbb{R}$? Is this series almost convergent on $(0, +\infty)$?

(iii) Let us consider the space with the measure: $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \mu)$, where $\mathcal{B}(\mathbb{R}_+)$ is a Borel σ-field of $\mathbb{R}_+$, and measures $\mu, \delta_n : \mathcal{B}(\mathbb{R}_+) \to [0, \infty)$ are defiined as

$$\mu(A) = \sum_{n=1}^{\infty} n\, \delta_n(A), \qquad \text{where} \qquad \delta_n(A) = \begin{cases} 1 & \text{for } n \in A \\ 0 & \text{for } n \notin A \end{cases},$$

for every set $A \in \mathcal{B}(\mathbb{R}_+)$. Compute

$$\int_{\mathbb{R}_+} \frac{x^2}{(1+x^2)^{t+1}} \mu(dt).$$

(iv) Compute

$$\sum_{n=0}^{\infty} \frac{\cos(nx)}{n!}.$$

Is this sereis uniformly convergent on $\mathbb{R}$?

2. **Linear algebra**

1. For a linear mapping of linear spaces $f : V \to W$ let $im(f) = f(V)$ denote the image. In the space of linear endomorphsms $End(\mathbb{R}^3)$ consider the subset

$$\Sigma_2(\mathbb{R}^3) = \{f \in End(\mathbb{R}^3) \mid \dim im(f) = 2\}.$$

(i) Prove that for every $f, g \in \Sigma_2(\mathbb{R}^3)$ we have $1 \leq \dim im(f \circ g) \leq 2$.

(ii) Give examples of endomorphisms $f_i, g_i \in \Sigma_2(\mathbb{R}^3)$, $i = 1, 2$, such that

$$\dim im(f_i \circ g_i) = i\,.$$

(iii) Let V be a finite dimensional vector space over $\mathbb{C}$. For an integer k we define the set

$$\Sigma_k(V) = \{f \in End(V) \mid \dim im(f) = k\}.$$

For fixed integers $0 \leq k \leq l \leq \dim V$ find a set

$$\{m \in \mathbb{N} \mid \exists f \in \Sigma_k(V),\ \exists g \in \Sigma_l(V),\ f \circ g \in \Sigma_m(V)\} \subset \mathbb{N}.$$

(iv) Does the answer in (iii) depend on the base field? Can one replace $\mathbb{C}$ by a field of arbitrary characteristic?

3. **Algebra**

Let $\mathbb{R}$ be a field of real numbers and let $I = (x^2 + x + 1)$ be an ideal in $\mathbb{R}[x]$.

(i) Prove that the ring $\mathbb{R}[x]/I$ is a field.

(ii) Write $z = (x^4 + x^3 - 2x + 1) + I$ as $a + bx + I$, $a, b \in \mathbb{R}$. Find z^{-1}..

(iii) Is the ring $\mathbb{Z}_7[x]/(x^2 + x + 1)$ a field?

(iv) If the answer in (iii) is negative, then find a prime p (if exists), for which the ring

$$\mathbb{Z}_p[x]/(x^2 + x + 1)$$

is a field.

4. **Topology**

Let X and Y be topological Hausdorff spaces.

(i) Let $A \subset Y$ be a compact subset. Prove that the quotient space Y/A is also Hausdorff.

(ii) Assume , that spaces X and Y are compact, $x_0 \in X$, and $A \subset Y$ is a non-empty closed subset. Prove that if spaces $X \setminus \{x_0\}$ i $Y \setminus A$ are homeomorphic, then the space X is homeomorfic to the space Y/A.

(iii) Find an example that (ii) does not hold if either X or Y are not compact.

(iv) Let D^2 be a unit disc in $\mathbb{R}^2$, and S^1 be its border. Prove that the quotient space D^2/S^1 is homeomorphic to the sphere S^2.

5. **Ordinary differential equations**

Let us consider an ordinary differential equation

$$ax'' + b(x,t)x' + c(x,t) = 0, \tag{1}$$

with $a \in \mathbb{R}$ and functions $b, c : \Omega \to \mathbb{R}$, where Ω is an open set such that $\Omega \subset \mathbb{R} \times \mathbb{R}_+$.

(i) Find a solution of an initial value problem (1) together with $x(1) = 1$, where

$$a = 0, \quad b(x,t) = t, \quad c(x,t) = -x \ln\left(\frac{x}{t}\right).$$

(ii) Assume that $a \neq 0$. Specify possibly general assumption that has to be imposed on functions b and c to assure that second order differential equation (1) with initial conditions $x(0) = A$ i $x'(0) = B$ has a unique solution.

(iii) Give an example of the problem (1) having at least one solution that which cannot be extended on $\mathbb{R}_+$. Explain your choice.

(iv) Examine stability of stationary solution of the problem (1), where

$$a = 1; \quad b(x,t) = b,\ b > 0; \quad c(x,t) = cx^3,\ c > 0.$$

6. **Probability**

(i) We toss a symmetrical coin 3 times. We know that an odd number of heads has been appeared. What is the probability that exactly 3 heads have been obtained?

(ii) We toss n times a symmetrical coin. What is the minimum number of n tosses so that the chance that a head will appear at least once is greater than $0,999$?

(iii) We draw the points of the plane. The drawn point is a random variable with a two-dimensional normal distribution with the expected value vector equal to $(0,0)$ and the covariance matrix $\Sigma = Diag(1,1)$. Find the density of the random variable and calculate the average distance of the drawn point from the origin of the coordinate system.

(iv) Let (X,Y) be a uniformly distributed random variable on

$$\Omega := \{(x,y) : 0 \leq x \leq y \leq 1\}.$$

Find the conditional density of X given $Y = y$. Calculate the conditional expectation of X given $Y = y$.

7. **Analytic functions**

Let $\mathbb{R}_- = \{t \in \mathbb{R} \mid t < 0\}$. For $z \in \mathbb{C}$ let $\mathrm{Re}(z)$ denote the real part. Let $Log : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$ be the principal branch of the logarithm.

(i) Find the real and imaginary part of $f(z) = (z-1)\mathrm{Re}(Log\, z)$.

(ii) In which points the funtion f satisfies the Cauchy-Riemann equations?

(iii) Check that $g(z) = z(Log\, z - 1)$ is an integral of $Log\, z$ and find the integral $\int_\gamma Log\, z\, dz$, where $\gamma : [0,1] \to \mathbb{C}$ is a curve given by $\gamma(t) = t - i(1-t)$.

(iv) Find the integral $\int_\gamma \overline{Log\, z}\, dz$, where γ is a curve defined in (iii).

8. **Functional analysis**

Let $\ell^p := \{x = (x_1, x_2, ...) : \|x\|_p := (\sum_{i=1}^{\infty} |x_i|^p)^{1/p} < \infty\}$ be a space of sequences endowed with the norm $\|\cdot\|_p$, $1 < p < \infty$.

(i) Show that the ball $B := \{x \in \ell^3 : \|x\|_3 \le 1\}$ is not compact.

(ii) Prove that for every $x \in \ell^{3/2}$ and $y \in B$ there holds

$$\sum_{i=1}^{\infty} x_i y_i \le \|x\|_{3/2}.$$

(iii) Let $A(x) = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, ..., \frac{1}{n}x_n, ...)$. Show that $A : \ell^2 \to \ell^2$ is a linear, bounded and compact operator.

(iv) Let $T : \ell^2 \to \ell^2$ be a self-adjoint operator (linear and bounded) and positive definite, i.e. there is a constant $c > 0$ such that

$$\langle Tx, x \rangle \ge c\|x\|_2^2 \quad \text{for every } x \in \ell^2,$$

where $\langle x, y \rangle := \sum_{i=1}^{\infty} x_i y_i$ for $x, y \in \ell^2$. Prove that for every $y \in \ell^2$ and every number $\lambda > -c$ there exists exactly one $x \in \ell^2$ such that $T(x) + \lambda x = y$. Show that every eigenvalue of the operator T is greater or equal to c. Find an example of a nonzero self-adjoint operator $T : \ell^2 \to \ell^2$ that has no nonzero eigenvalues.

9. **Programming languages**

(a) The following program in C:

```
#include <stdio.h>
#define SUM(a,b) a+b
int main()
{
    int x = 2; int y = 3;
    printf("%d",SUM(x,y)*5);
    return 0;
}
```

does not ouput the number 25, but some other number. What is that number? Why is that?

(b) Describe the set of those pairs of initial values of integer variables `x` and `y` for which the following loop in C terminates:

```
while (x>y) y+=(--x)
```

You may assume that variables can take arbitrary integer values.

(c) Give the types of functions `f` i `g` defined by the following sequence of declarations in OCaml:

```
let x:int = 5
let f y = y + x
let g z = z f + x
```

(d) In Haskell, assume that we have a function

```
plus :: [Int] -> [Int] -> [Int]
```

which adds corresponding elements of two lists of integers, for example:

```
plus [1,2,3] [4,5,6] = [5,7,9].
```

Give the first ten elements of the list `s` defined by the expression:

```
let s = 0:1:(plus s (tail s))
```

What is this sequence of numbers?

10. **Discrete mathematics**

In this problem we consider nonnegative integer numbers in the decimal representation (without leading zeroes). Let n be a positive integer.

(a) How many are there n-digit numbers such that the product of all their digits is divisible by 10?

(b) How many are there n-digit numbers such that the product of all their digits gives the remainder 3 when divided by 5?

(c) How many are there pairs of n-digit numbers (a, b), such that no carry occurs in the standard pencil-and-paper addition of a and b?

(d) How many numbers are there, with at most n digits, that contain the subsequence 1, 2?

In all cases you should provide either an explicit formula or a detailed description of a way to derive such a formula.

11. **Algorithms and data structures**

We are given on input an array that represents a sequence of n numbers from the set $\{-10, \ldots, 10\}$. For each of the following problems, propose a linear-time algorithm.

(a) For a number k given on input, choose a subsequence of k elements with the maximal sum.

(b) For an even number k given on input, choose a subsequence of k elements with the maximal product.

(c) Check if the sequence contains a *consecutive* subsequence with sum 0.

(d) Check if the sequence contains a subsequence with sum 0.

12. **Logic** In this problem we consider first-order logic with equality, with no additional relation or function symbols. For each property below, define a set of sentences with this property or justify why such a set does not exist:

(a) a finite set of sentences that has models, but only finite ones,

(b) a finite set of sentences that has models, but only infinite ones,

(c) a set of sentences that has arbitrarily large finite models, but no infinite ones,

(d) a set of sentences that has a model with n elements if and only if n is a prime number.

13. **Automata and formal languages**

For any finite group $\mathbf{G} = (G, \cdot, 1)$, let us define a function $\Pi_{\mathbf{G}} : G^* \to G$ by induction:

$$\Pi_{\mathbf{G}}(\epsilon) = 1 \qquad \Pi_{\mathbf{G}}(aw) = a \cdot \Pi_{\mathbf{G}}(w) \quad \text{dla } a \in G, w \in G^*,$$

and let us define a language:

$$L_{\mathbf{G}} = \{w \in G^* \colon \Pi_{\mathbf{G}}(w) = 1\}.$$

Justify the following statements:

(a) For every finite group $\mathbf{G}$, the language $L_{\mathbf{G}}$ is regular.

(b) For every finite group $\mathbf{G}$, every deterministic finite automaton that recognizes $L_{\mathbf{G}}$ has at least $|G|$ states.

(c) For every finite group $\mathbf{G}$, every *non*deterministic finite automaton that recognizes $L_{\mathbf{G}}$ has at least $|G|$ states.

(d) Let $\mathbf{G} = \mathbf{S}_n$ be the group of all permutations of an n-element set. Then there is a nondeterministic finite automaton with $\mathcal{O}(n^2)$ states that recognizes the language $G^* - L_{\mathbf{G}}$.

14. **Computational complexity**

For any language $L \subseteq \Sigma^*$, let us define:

$$L^{\exists} = \{v \in \Sigma^* \colon \exists w \in \Sigma^*(|w| = |v| \wedge vw \in L)\},$$
$$^{\exists}L = \{v \in \Sigma^* \colon \exists w \in \Sigma^*(|w| = |v| \wedge wv \in L)\}.$$

Justify the following statements:

(a) If L in in **PTIME** then $L^{\exists}$ and $^{\exists}L$ are in **NP**.

(b) If L in in **PSPACE** then $L^{\exists}$ and $^{\exists}L$ are in **PSPACE**.

(c) There is a language L in **PTIME** such that $L^{\exists}$ and $^{\exists}L$ are **NP**-complete.

(d) There is an **NP**-complete language L such that $L^{\exists}$ and $^{\exists}L$ are in **PTIME**.

15. **Concurrent programming**

In shared-memory architectures, in which multiple threads can perform read and write operations on shared variables, a popular consistency model for accessing such variables is *sequential consistency.* Consider the following program, which consists of three threads, `t1`, `t2` and `t3`, performing read and write operations on three shared variables, `x`, `y` i `z`.

```
int x = 0, y = 0, z = 0;
void t1() {              void t2() {              void t3() {
  x = 1;                   y = 1;                   z = 1;
  print(y);                print(x);                print(x);
  print(z);                print(z);                print(y);
}                        }                        }
```

All threads are activated at roughly the same time and variable accesses are sequentially consistent. The function `print` outputs the value of its argument, and its every execution is atomic, i.e. after its argument is computed and before it is output no other operations are performed.

Under these assumptions, which of the following statements are true?

(a) The character sequence `001011` is a correct output of the program.

(b) The character sequence `001101` is a correct output of the program.

(c) The character sequence `110011` is a correct output of the program.

(d) There exists exactly one character sequence which begins with `000` and is a correct output of the program.

16. **Bioinformatics**

In an alignment problem, assume a reward $\alpha > 0$ for a match, a penalty $\beta \geq 0$ for a gap, and a penalty $\gamma \geq 0$ for a mismatch.

For a pair of sequences MIMUWAAMIMUW oraz MIMUWCCMIMUW:

(a) determine the optimal score of a global alignment depending on α, β i γ,

(b) characterize all possible optimal global alignments and count them,

(c) as in (a) and (b) but for local alignments, under the assumption that $\beta = \gamma$.