Warsaw Doctoral School of Mathematics and Computer Science

Entrance exam, July 3, 2020

On the following pages you will find 16 problems related to various areas of Mathematics and Computer Science. You are expected to choose and solve any 4 of them. Each problem is worth the same number of points.

You are free to choose any problems you wish, i.e., candidates for studies in Mathematics may choose also Computer Science problems and *vice versa*.

Every problem is composed of a few subproblems of comparable size, but each problem (i.e. all its subproblems) is graded as a whole.

You can attempt to solve more than 4 problems. All your solutions will be graded, but only 4 best-graded solutions will contribute to your general grade.

Every problem should be solved on a separate sheet of paper, signed with your full name and marked with the problem number.

Good luck!

1. **Linear algebra**

   Let $n > 1$ be a natural number. Consider the vector space $V = \mathbb{R}^n$ with the standard scalar product. The product of the vectors v and w is denoted by $\langle v, w \rangle$. Let $e_1,\ e_2, \ldots e_n$ be the standard basis. Let $f : V \to V$ be the linear map such that

   $$f(e_i) = \begin{cases} e_{i+1} - e_i & \text{if } i = 1, 2, \ldots n - 1 \\ e_1 - e_n & \text{if } i = n. \end{cases}$$

   Let $\Delta = f^* \circ f$, where f^* is the adjoint to f, i.e. for $v, w \in V$ we have $\langle f(v), w \rangle = \langle v, f^*(w) \rangle$.

   (a) For $n = 3$ find a basis of V consisting of eigenvectors of Δ.

   (b) For each $n > 1$ show that there exists a linear map $\pi : V \to V$ such that

   $$\pi \circ \Delta = \Delta \circ \pi, \qquad \pi \circ \pi = \pi$$

   and the image of π is equal to $ker(\Delta)$.

   (c) For each $n > 1$ show that all the eigenvalues of Δ are nonnegative real numbers.

   (d) For each $n > 1$ show that $ker(\Delta) = ker(f)$ and find a basis of $ker(\Delta)$.

2. **Algebra**

   Let $G = A_4$ be a group of all even permutation of the set $\{1, 2, 3, 4\}$.

   (a) Compute $|G|$.

   (b) Find a subgroup $H \subset G$ which is a non-trivial, proper, normal subgroup of G.

   (c) Let $K \subset G$ be a subgroup such that $|K| = 3$. Prove that K is not a normal subgroup of G. Give an example of such subgroup.

   (d) Find a minimal set of generators of G.

3. **Topology**

   Let M be the open Möbius strip, which we define as the quotient space

   $$M = [-10, 10] \times (-1, 1) / \sim,$$

   where

   $$(s, t) \sim (s', t') \quad \Leftrightarrow \quad \left((s = s' \ \wedge \ t = t') \ \vee \ (s = -10 \ \wedge \ s' = 10 \ \wedge \ t = -t') \right).$$

   (a) Let $\mathbb{Z}$ be the set of integer numbers. Does the product

   $$\prod_{i \in \mathbb{Z}} M$$

   with the product topology (Tikhonov topology) contain a countable dense set?

(b) Show that M contains a subspace S, which is homeomorphic to the circle S^1 such that $M \setminus S$ is connected.

(c) Compute the fundamental group $\pi_1(M)$.

(d) Let $N = \{[(s,t)] \in M : |t| \geq \frac{1}{2}\}$. Consider the quotient space $P = M/N$. Show that for any two points $x, y \in P$ there exists a homeomorphism $\phi : P \to P$ such that $\phi(x) = y$.

4. **Probability**

Random variable X has the geometric distribution

$$\mathbb{P}(X = x) = (1 - p)p^x, \qquad x = 0, 1, 2, \ldots,$$

for some $0 < p < 1$.

(a) Determine the probability that $X \geq 10$.

(b) Determine the probability that X is an even number. Is it greater than the probability of getting an odd number?

(c) Calculate the expectation $\mathbb{E}X$.

(d) Calculate the expectation of X under the condition that X is even. Is it greater than the unconditional expectation $\mathbb{E}X$?

5. **Analysis**

Let $Q = \{x \in \mathbb{R}^2 : ((x_1 - 1)^2 + x_2^2)((x_1 + 1)^2 + x_2^2) = 1\}$
and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function.

(a) Does f attain its global maximum and minimum on Q? Substantiate your claim.

(b) At which points $(\bar{x}_1, \bar{x}_2) \in Q$ do there exist a continuous function h defined on a neighbourhood N_1 of $\bar{x}_1$ and a neighbourhood N_2 of $\bar{x}_2$ such that $Q \cap (N_1 \times N_2) = \{(x_1, h(x_1)) : x_1 \in N_1\}$?
At which points every such function h is differentiable?

(c) Describe the method of Lagrange multipliers. Formulate a relevant theorem. At which points of Q the assumptions of your theorem are satisfied?

(d) Let $f = x_1^2 - x_2^2$. Calculate the maxima and minima of f on Q (Hint: using Lagrange multipliers yields the result quite fast)

6. **Analytic functions**

Let $F(z) = \frac{1}{z^2 + z + 1} \in \mathbb{C}(z)$ be a rational function.

(a) Find all poles of $F(z)$ in the complex domain.

(b) Compute all residues of $F(z)$ in the complex domain.

(c) Let $\Gamma = \{z \in \mathbb{C} : |z| = 5\}$. Compute $\int_\Gamma F(z)dz$, where Γ has the standard orientation.

(d) Compute $\int_{-\infty}^{\infty} F(x)dx$ (here we treat F as a real function).

7. **Functional analysis**

Let $V = L_0^2([0, \pi])$ be the space of square integrable functions on the interval $[0, \pi]$ and let V_0 be the subspace consisting of the function with mean value 0. The operator $T : V_0 \to V$ is given by the formula $Tf(x) = \int_0^x f(t)dt$.

(a) Show that T is a bounded operator. Find the norm of T.

(b) Find an isometry $S : V \to V_0$ such that ST is a selfadjoint operator.

(c) Do there exist continuous operators $A : V_0 \to C([0, \pi])$ and $B : C([0, \pi]) \to V$ such that $T = BA$? Here $C([0, \pi])$ denotes the Banach space of continuous functions on $[0, \pi]$ with the supremum norm. Is T a compact operator?

(d) Let $n > 0$ be an integer. Let $\rho(n)$ be the infimum of the set of norms of the operators $R : V \to V_0$, such that such that RT is an isometry on some n-dimensional subspace $W \subset V_0$. That is,

$$\rho(n) = \inf_{R:\, V \to V_0} \{\|R\| \; : \; \exists W \subset V \quad \dim(W) = n, \quad \forall f \in W \quad |RT(f)| = |f|\}.$$

(here $\| \cdot \|$ denotes the operator norm and $| \cdot |$ denotes the L^2 norm in the space of functions). Compute $\rho(n) \leq n$.

Hint: In the above problems use the orthogonal bases consisting of the functions $f_n(t) = \cos(nt)$ and $g_n(t) = \sin(nt)$, $n = 1, 2, 3, \ldots$.

8. **Differential equations**

(a) Consider a function $f : \mathbb{R} \to \mathbb{R}$, differentiable, bounded, with bounded derivative and with exactly three points at which it is zero: $x_1 < x_2 < x_3$, with $f(x) \geq 0$ if and only if $x \in [x_1, x_3]$, as drawn on the sketch.
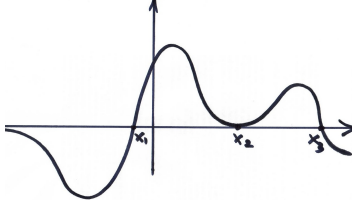
Consider the ordinary differential equation

$$\dot{x}(t) = f(x(t))$$
with the initial condition $x(0) = x_0$.

$$(1)$$

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What are the equilibria (i.e. steady states), and which of them are Lyapunov stable?
Does there exist x_0 such that $\lim_{t\to+\infty} x(t) = +\infty$?
Does there exist x_0 such that $\lim_{t\to+\infty} x(t) = -\infty$?
Which of the equilibria can appear as $\lim_{t\to+\infty} x(t)$ for nonconstant x?

(b) Formulate the Picard-Lindelöff Theorem about the existence and uniqueness of solutions of ordinary differential equations. Can it be applied to Eq. (1)?

(c) Consider $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ and the ordinary differential equation in $\mathbb{R}^2$

$$\dot{x}(t) = Ax(t) + b$$
$$\text{with the initial condition } x(0) = x_0, \tag{2}$$

where $b \in \mathbb{R}^2$ is a constant vector. Without solving, find the equilibrium and answer what kind of equilibrium it is (sink/stable node, saddle, etc.)?

Calculate the set of all initial conditions x_0 for which the trajectory of x converges to the equilibrium.

(d) Consider the differential equation

$$\dot{x}(t) = Cx(t), \tag{3}$$

where C is a 2×2 real matrix.

Let y_1 and y_2 be two solutions of Eq. (3) such that $y_1(0)$ and $y_2(0)$ are linearly independent
Define the matrix

$$M(t) = [y_1(t), y_2(t)].$$

Consider the differential equation

$$\dot{x}(t) = Cx(t) + B(t)$$
$$\text{with the initial condition } x(0) = x_0, \tag{4}$$

where $B : \mathbb{R} \to \mathbb{R}^2$ is a continuous function. Express the solution of the differential equation Eq. (4) in terms of $M(t)$, $B(t)$ and x_0.

9. **Programming languages**

(a) Give the types of functions `f` and `g` defined by the following sequence of declarations in OCaml:

```
let x:int = 4
let f a b = a + b + x
let g = f 1
```

(b) Give the weakest logical condition ϕ that guarantees correctness of the following statement in Hoare's logic for proving correctness of programs that operate on integer variables (with syntax taken from C):

$$\{\phi\}$$
```
if (x>y) z = x-y; else z = y-x;
```
$$\{z = y\}$$

(c) Give the first 10 elements of the list x defined in Haskell by:

```
let x = 0:1:(map (\n -> 1-n) x)
```

(d) For the following classes in Java:

```
class A {
    void f() {g();h();}
    void g() {System.out.print("1");}
    void h() {System.out.print("2");}
}
class B extends A {
    void h() {System.out.print("3");}
}
```

what will be the result of executing the following statements?

```
A a = new A(); A b1 = new B(); B b2 = new B();
a.f(); b1.f(); b2.f();
```

And what would be the result if all methods in both classes were declared with the keyword `static`? Justify your answers briefly.

10. **Discrete mathematics**

Consider $n \geq 0$ beads.

Assume first that every bead has a different color.

(a) In how many ways one can choose an even number of beads?

(b) In how many ways one can choose at most $n/2$ beads?

Now assume that n is divisible by 3 and the i-th bead's color is i mod 3.

(c) In how many ways one can choose k beads, where $0 \leq k \leq n/3$?

(d) More generally, in how many ways one can choose k beads, where $0 \leq k \leq n$?

Two choices of beads are considered different if they differ in the number of chosen beads of some color. In all cases an explicit (compact) formula is expected.

11. **Algorithms and data structures**

Consider n positive integers that represent masses of particles. Two particles of the same mass can merge into a single particle, with the mass equal to the sum of their masses. A set of particles is called stable if no two particles in it can merge. Particles of the same mass are considered indistinguishable. We assume that the cost of arithmetic operations is $O(1)$.

(a) Prove that from every set of particles, only one stable set can be obtained.

(b) Propose a data structure that allows insertions of particles to an initially empty set, and after each such operation outputs the size of the stable set corresponding to the current set of particles. The amortized cost of an operation on the structure, in the case of n insertions, should be $O(\log n)$.

(c) Propose a (deterministic) linear-time algorithm which for a given set of particles computes the resulting stable set, assuming that the masses of particles are given in the non-decreasing order.

(d) Propose a (deterministic) linear-time algorithm which for a given set of n particles computes the resulting stable set, assuming that the masses of original particles come from the interval $\{1, \ldots, n^2\}$.

12. **Logic**

Give sentences in first-order logic (with equality, over a signature of your choice with relation and/or function symbols) with the following properties, or argue why such sentences do not exist:

(a) a sentence which has some models but only two-element ones,

(b) a sentence which has arbitrarily big finite models but only of even size;

(c) a sentence which has some models but only infinite ones,

(d) a sentence which has arbitrarily big finite models but no infinite models.

13. **Automata and formal languages**

Consider the function $\oplus : \{0,1\}^2 \to \{0,1\}$ defined as follows:

$$0 \oplus 0 = 0 \qquad\qquad 1 \oplus 1 = 0 \qquad\qquad 1 \oplus 0 = 1 \qquad\qquad 0 \oplus 1 = 1$$

We extend it to words over the alphabet $\{0,1\}$ so that for every $u, w \in \{0,1\}^*$ and $a, b \in \{0,1\}$:

$$w \oplus \varepsilon = w \qquad\qquad \varepsilon \oplus w = w \qquad\qquad ua \oplus wb = (u \oplus w)(a \oplus b)$$

For example, $000000111 \oplus 1010 = 000001101$. Furthermore, for two languages K, L we define $K \oplus L$ as $\{k \oplus l \mid k \in K, l \in L\}$.

(a) Find all languages $K \subseteq \{0,1\}^*$ such that for every $L \subseteq \{0,1\}^*$:

$$L = (L \oplus K) \oplus K$$

(b) Write a regular expression for the language $0^* \oplus 1^* \oplus 1^* \oplus 1^*$.

(c) Prove that for every regular language R and a word $w \in \{0,1\}^*$, the language $R \oplus \{w\}$ is regular.

(d) Prove that the language $L = \{1 \cdot 0^n \cdot 1^{2n} \mid n \in \mathbb{N}\} \oplus \{1 \cdot 0^{2m} \cdot 1^m \mid m \in \mathbb{N}\}$ is not context-free.

14. **Computational complexity**

Let $\overline{\phi}$ denote a standard representation of a formula of propositional logic ϕ as a word over the alphabet $\Sigma = \{0, 1, \vee, \wedge, \neg, (,)\}$ where propositional variables are represented as binary numbers. Consider the language over the alphabet $\Sigma \cup \{\$\}$:

$$L = \{\overline{\phi}\$\overline{\psi} : \phi \text{ i } \psi \text{ are not equivalent}\}$$

Justify the following statements:

(a) L is **NP**-complete;

(b) L, intersected with the language of those words where at most 2020 different propositional variables appear, is in the class **PTIME**;

(c) L, intersected with the language of those words where neither negation ($\neg$) nor conjunction ($\wedge$) occurs, is in the class **LOGSPACE**;

(d) if the complement of L is in **NP** then **NP**=**coNP**.

15. **Computer systems**

Consider a system with memory paging with page size 4096B, where each entry (i.e. an address) in a page table takes 8B, and a page in a page table at any level also takes 4096B.

(a) Does such a system allow for mapping virtual addresses 65539 i 131331 to the same physical address?

(b) Does such a system allow for mapping virtual address 65539 to two different physical addresses 65539 and 135171 (not necessarily in the same process)?

(c) How many levels does this paging system have to provide to enable managing 1MB of memory?

(d) How many levels does this paging system have to provide to enable managing 1GB of memory?

Justify your answers.

16. **Concurrent programming**

In shared-memory architectures, in which multiple threads can perform read and write operations on shared variables, a popular consistency model for accessing such variables is *sequential consistency.* Consider the following program, which consists of three threads, t1, t2, and t3, performing read and write operations on three shared variables, x, y, and z.

```
int x = 0, y = 0, z = 0;
void t1() {                  void t2() {                  void t3() {
  int a = y + 1;               int b = z + 1;               int c = x + 1;
  x = a;                       y = b;                       z = c;
  printf("%d%d",y%2,z%2);      printf("%d%d",z%2,x%2);      printf("%d%d",x%2,y%2);
}                            }                            }
```

All threads are activated at roughly the same time, variable accesses are sequentially consistent, the arguments for the function printf are computed from the last (rightmost) to the first (leftmost), characters printed by a single call to printf appear in the output stream as a consecutive string, and besides the characters printed by the calls to printf in the three threads, the program does not produce any other output. Under these assumptions, decide which of the following statements are true.

(a) The character sequence 001110 is a correct output of the program.

(b) The character sequence 111101 is a correct output of the program.

(c) The condition 0 <= x <= 2 is an invariant of the program.

(d) The condition 0 <= x + y + z <= 6 is an invariant of the program.

Justify your answers. More precisely, for each item (a)–(d), your answer must begin with a single word, either "YES" or "NO", followed by a justification of that answer.