Doctoral School of Exact and Natural Sciences
physical sciences – written examination

Instructions. In your solutions, present the reasoning leading to the result. Write the final numerical results to 3 or 2 significant figures, after appropriate rounding, e.g. $1.23456 \cdot 10^{-19} \approx 1.23 \cdot 10^{-19}$.

The examination problems are printed on four pages.
Problems 1–14 are easier problems. Submit or send solutions to only **four** of these problems. For each solution you may obtain up to 6 points.
Problems 15–21 are harder problems. Submit or send solutions to only **two** of these problems. For each solution you may obtain up to 8 points.

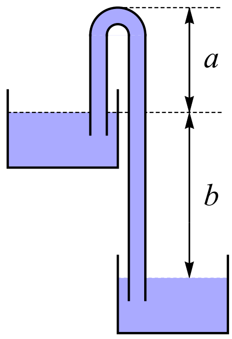
The solution to each problem must be written on a separate sheet. Submitting or sending a larger number of problems than specified for a given range will result in only the problems graded **lowest** being included in the score.

EASIER PROBLEMS

Problem 1

A space station has the shape of a cylindrical shell of radius $R = 100\,\mathrm{km}$ rotating about the axis of the cylinder with such an angular velocity ω that the astronauts staying in the station can enjoy an apparent gravity with acceleration $g = 10\,\frac{\mathrm{m}}{\mathrm{s}^2}$, which is in fact the effect of being in a non-inertial frame. A bored astronaut lay down on the floor of a high room and spat "upward", i.e. toward the axis of the cylinder, with the speed of the saliva drop equal to $v = 10\,\frac{\mathrm{m}}{\mathrm{s}}$. Will the astronaut spit on himself, or will the saliva drop fall next to him? Justify your answer with calculations.

Problem 2

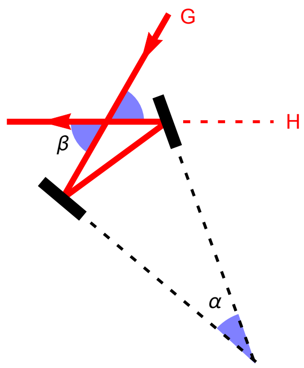


The figure next to this text shows a siphon used to transfer liquid between vessels at different levels. What is the maximum height a above the surface of the upper vessel at which the upper end of the siphon may be located when diethyl ether is being siphoned under a laboratory fume hood, if the difference in the liquid levels in the vessels is initially fixed and equal to $b = 50\,\mathrm{cm}$? The siphon tube has the same diameter along its entire length, the length of the curved section of the siphon is very small, and the liquid being transferred is pure and contains no dissolved gases. The whole system is in a constant and uniform gravitational field of strength $g = 9.81\,\mathrm{m/s^2}$ and at a constant temperature $T = 20°\mathrm{C}$. The saturated vapor pressure of diethyl ether at this temperature is $p_n = 586\,\mathrm{hPa}$, and its density is $\rho = 714\,\mathrm{kg/m^3}$. The atmospheric pressure is $p_0 = 1013\,\mathrm{hPa}$. Neglect the viscosity of the liquid and flow resistance. Diethyl ether was formerly used as an anesthetic and as a narcotic.

Problem 3

A stationary nucleus of the fluorine isotope ${}^{18}_{9}\mathrm{F}$ undergoes radioactive decay via a β^+ transformation. During the decay of this fluorine nucleus, the products are: the nucleus of some element, a positron, and an electron neutrino. One decay of the fluorine nucleus occurred in such an interesting way that the momenta of the positron and the neutrino had the same magnitude and direction. Determine the speed of the positron. The neutrino has zero electric charge, and its mass may be neglected. The mass of the fluorine nucleus is $m_\mathrm{F} = 17.99600\,\mathrm{u}$, the mass of the resulting nucleus is $m_\mathrm{X} = 17.99477\,\mathrm{u}$, and the positron has mass $m_e = 0.00055\,\mathrm{u}$, where u is the atomic mass unit, approximately equal to $931.5\,\mathrm{MeV}/c^2$, and c is the speed of light in vacuum. The kinetic energy of the resulting nucleus is negligibly small.

Problem 4



A sextant, shown schematically in the figure next to this text, is a popular navigational instrument used to measure the altitude of celestial bodies. It consists of two movable plane mirrors, one of which is partially transparent and equipped with a scale allowing the angle between them to be measured. The sextant works as follows: by observing the horizon line H through the partially transparent mirror and changing the angle α between the mirrors, we superimpose the image of the star G on the horizon line. The altitude of the star β above the horizon is equal to twice the angle α between the mirrors. Justify this fact.

Problem 5

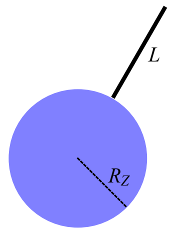
A particle of spin $\frac{1}{2}$ is in a constant, uniform magnetic field $\vec{B} = (0, 0, B)$. The Hamiltonian of the system has the form $\hat{H} = -\mu B \sigma_z$, where $\sigma_z = \mathrm{diag}(1, -1)$, and μ is the magnetic moment. The system is in thermal equilibrium with a reservoir at temperature T; the equilibrium state is described by the density matrix $\hat{\rho} = Z^{-1} e^{-\beta \hat{H}}$, where $Z = \mathrm{Tr}(e^{-\beta \hat{H}})$ is the partition function and $\beta = (k_B T)^{-1}$, while k_B is the Boltzmann constant. Determine the eigenvalues of $\hat{H}$ and calculate the partition function Z. Determine the density matrix in the basis of eigenstates of σ_z.

Calculate the mean energy of the system $U = \mathrm{Tr}(\hat{\rho}\hat{H})$ and the magnetization $M = \mu\langle\sigma_z\rangle$. Discuss the limits $T \to 0$ and $T \to \infty$.

Problem 6
A tube of length $L = 20\,\mathrm{cm}$ was plugged at one end with a finger and then inserted into water in such a way that the tube was vertical and the end plugged with the finger was just above the surface of the water. Then the finger was quickly removed, so a certain amount of water entered the tube and spurted out above the liquid surface. To what height above the surface did the water spurt from the tube? The whole system is in a constant and uniform gravitational field of strength $g = 9.81\,\mathrm{m/s^2}$. The density of water is $\rho = 1000\,\mathrm{kg/m^3}$. Water is an incompressible liquid. The viscosity of water and air resistance may be neglected.

Problem 7
You have at your disposal: (i) a wooden skewer, (ii) a piece of string, (iii) a pen, (iv) graph paper. Determine the value of π as accurately as possible. Estimate the uncertainty of the obtained result. (Problem unavailable for an online exam.)

Problem 8
One of Richard Heinlein's science-fiction novels describes an Earth "satellite" in the shape of a very strong, homogeneous rope of constant cross-section. The rope is arranged along the Earth's radius, and its lower end remains at all times at a fixed point on the Earth's surface, as shown in the figure next to this text. What would the length of the rope have to be for it to remain in such a position? The radius of the Earth is $R_Z = 6400\,\mathrm{km}$, and Newton's gravitational constant is $G = 6.67 \cdot 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$.



Problem 9
A lithium nucleus has four spin states corresponding to the quantum numbers $m = -\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}$. When lithium is in a magnetic field of induction B, the energies of these states are $E_m = -m\mu B$, where $\mu = 1.03 \cdot 10^{-7}\,\mathrm{eV/T}$. What is the probability of finding the lithium nucleus in the state with quantum number $m = -\frac{1}{2}$, if a lithium sample is placed in a field $B = 0.7\,\mathrm{T}$ at temperature $T = 20°\mathrm{C}$? The Boltzmann constant is $k_B = 8.62 \cdot 10^{-5}\,\mathrm{eV/K}$.

Problem 10
In a certain experiment, particles of dark matter were searched for. The experiment operated for one year and did not observe any particle. What is the upper bound on the number of particles λ detected during one year in this experiment at the confidence level CL = 95%? The probability distribution of the number n of registered particles is given by the Poisson distribution $P(n;\lambda) = e^{-\lambda}\lambda^n/n!$.

Problem 11
Data collected in a certain experiment are stored in a text file named `dane.txt`, as one column of real numbers from a range that is unknown, but the numbers are available on the given computer. The number of rows in the file is also unknown. Using any programming language or pseudocode, write the algorithm of a program that reads the data from the file and finds the largest number contained in it.

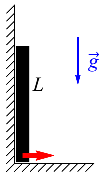
Problem 12
The wavenumber corresponding to the fundamental vibrational transition ($0 \to 1$) in a chlorine molecule Cl_2 is $\tilde{\nu} = \frac{1}{\lambda} = 559.7\,\mathrm{cm^{-1}}$, where λ is the wavelength corresponding to this transition. Calculate the ratio of the number of Cl_2 molecules in the ground state to the number of molecules in the first excited vibrational state at temperature $T = 298\,\mathrm{K}$. At what temperature is this ratio equal to 5? Assume that the Cl_2 molecules are in thermal equilibrium. Take the Boltzmann constant to be $k_B = 1.38 \cdot 10^{-23}\,\mathrm{J/K}$ and the Planck constant to be $h = 6.63 \cdot 10^{-34}\,\mathrm{J\,s}$.

Problem 13
Lysozyme, a protein with molar mass $M = 14\,300\,\mathrm{g/mol}$, forms a crystal with density $\rho = 1.24\,\mathrm{g/cm^3}$ and the following unit-cell parameters: $a = 30.08\,\text{Å}$, $b = 57.05\,\text{Å}$, $c = 68.52\,\text{Å}$ and $\alpha = \beta = \gamma = 90°$. The unit cell contains four protein molecules. The crystal also contains water molecules. Estimate the water content (mass percent) in the crystal, neglecting the presence of salts and organic compounds used during crystallization. The crystal was illuminated with an X-ray beam of wavelength $\lambda = 1.54\,\text{Å}$. At what angle with respect to the incident beam direction will the first-order diffraction maximum appear for reflection from the family of planes (100)? Take Avogadro's constant to be $N_A = 6.022 \cdot 10^{23}\,\mathrm{mol^{-1}}$.

Problem 14
The dissociation constant K_D of a certain protein–ligand complex with 1:1 stoichiometry is $6.7\,\mu\mathrm{M}$. Calculate what fraction (percent) of the protein is bound to the ligand in thermodynamic equilibrium in a mixture prepared by mixing a protein solution of concentration $c_{B0} = 30\,\mu\mathrm{M}$ with a ligand solution of concentration $c_{L0} = 1.0\,\mathrm{mM}$ in a volume ratio of 9:1. In what volume ratio should these solutions be mixed so that, at equilibrium, 99% of the protein forms a complex with the ligand? It is useful to remember that $1\,\mathrm{M} = 1\,\mathrm{mol/dm^3}$.

HARDER PROBLEMS

Problem 15



A homogeneous, thin rod of mass m and length L stands vertically on a smooth floor next to a wall with a smooth surface. The lower end of the rod was gently pushed and began to move freely away along a plane perpendicular to the wall. At what angle will the rod be inclined to the floor at the instant when the upper end of the rod separates from the wall? The moment of inertia of the rod with respect to a perpendicular axis passing through its center of mass is $I = \frac{1}{12}mL^2$.

Problem 16

In the nuclear processes taking place inside the Sun, electron neutrinos ν_e, called solar neutrinos in this context, are produced. They were first registered in 1968 in the Homestake mine in the USA, in an experiment carried out by Raymond Davis. The detector used in this experiment was filled with tetrachloroethylene C_2Cl_4 and detected the reaction

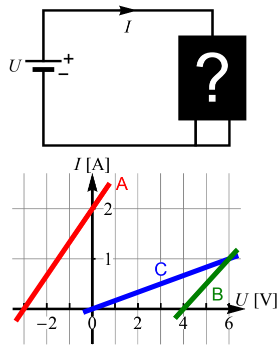
$$\nu_e + {}^{37}_{17}\mathrm{Cl} \longrightarrow {}^{37}_{18}\mathrm{Ar} + e^- \,.$$

Determine the minimum neutrino energy for which this reaction can occur. Assume that the ${}^{37}_{17}\mathrm{Cl}$ nucleus was at rest. The mass difference between the neutron and the proton is $1.293\,\mathrm{MeV}/c^2$, the electron mass is $0.511\,\mathrm{MeV}/c^2$, where c is the speed of light in vacuum. The binding energy $B(A,Z)$ of a nucleus with atomic number Z and mass number A may be described by the liquid-drop model as

$$B(A,Z) = a_V A - a_S A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_A \frac{(A-2Z)^2}{A} + \frac{\delta}{A^{1/2}}\,,$$

where $a_V = 15.74\,\mathrm{MeV}$, $a_S = 17.62\,\mathrm{MeV}$, $a_C = 0.72\,\mathrm{MeV}$, $a_A = 23.43\,\mathrm{MeV}$, and $\delta = +a_{\mathrm{par}}, 0, -a_{\mathrm{par}}$, respectively, for even–even nuclei, nuclei with odd A, and odd–odd nuclei, while $a_{\mathrm{par}} = 15.56\,\mathrm{MeV}$.

Problem 17



A given electrical "black box" contains an unknown circuit consisting of resistors and DC current sources. The box has three terminals: A, B, and C. Using the setup shown in the upper figure next to this text, where the black box is marked by a rectangle with a question mark, the dependence of the current I on the applied voltage U was determined. These dependences were determined for all three nonequivalent ways of connecting the terminals and are shown in the lower figure next to this text. The positive direction of current flow is defined by the arrow in the figure, and the symbol next to each characteristic corresponds to the terminal that, in that case, was connected to the positive terminal of the source, while the two remaining terminals were shorted and connected to the negative terminal of the source. Give the simplest, i.e. containing the smallest number of elements, possible electrical circuit of the black box and determine the parameters of its elements.

Problem 18

When a type II supernova collapses, after an initial rapid increase in brightness there follows a gradual decrease, which may last for months. The light curve is then dominated by the energy released in the decays of radioactive isotopes produced during the collapse. One of the main isotopes responsible for the emission of radiation during this time is ${}^{56}\mathrm{Co}$, which decays to ${}^{56}\mathrm{Fe}$ with a half-life of $\tau_{1/2} = 77.7$ days, emitting on average energy $E = 3.7\,\mathrm{MeV}$ per decay. Determine the rate of change of the brightness of such a supernova and the change in its brightness over one year, if initially $M_{\mathrm{Co}} = 1.49 \cdot 10^{29}\,\mathrm{kg}$ of cobalt was produced, which is 7.5% of the mass of the Sun. The absolute magnitude M_g of a star is given by $M_g - M_\odot = -2.5\log_{10}(L_g/L_\odot)$, where $M_\odot$ is the absolute magnitude of the Sun, equal to 4.79 mag, while L_g and $L_\odot$ denote the radiated power of the star and the Sun, respectively.

Problem 19

Sounding the optical properties of the lowest layers of the atmosphere may be performed using a balloon of own mass m_B, which carries measuring instruments of total mass m_E to a height z above the Earth's surface. Assume that atmospheric air may be treated as a dry ideal gas whose temperature decreases with height with a constant gradient Γ. Determine the radius of a balloon that will float at $z = 3\,\mathrm{km}$, if its $m_B = 50\,\mathrm{kg}$ and $m_E = 30\,\mathrm{kg}$, when at the Earth's surface the temperature is $T_0 = 290\,\mathrm{K}$ and the pressure is $p_0 = 1013\,\mathrm{hPa}$. In the calculations use the dry adiabatic lapse rate $\Gamma = 0.01°\mathrm{C/m}$ and the specific gas constant (the ratio of the gas constant to molar mass) for dry atmospheric air $R_a = 287\,\mathrm{J\,K^{-1}kg^{-1}}$. The volume of the balloon envelope itself and of the measuring instruments is negligibly small. The own mass m_B also includes the mass of the lifting gas.

Problem 20

A solution of an oligonucleotide with sequence GCAGCTGC and concentration $c_M = 10\,\mu\mathrm{mol/dm^3}$ was prepared, and then its absorbance at wavelength 260 nm was measured at different temperatures, giving the values: $A_1 =$

0.521 at $T_1 = 10°\text{C}$ and $A_2 = 0.799$ at $T_2 = 40°\text{C}$. Absorption measurements were performed in a cuvette with optical path length $l = 1\,\text{cm}$. Calculate the standard enthalpy ΔH^0 and entropy ΔS^0 of the dimerization reaction of this oligonucleotide, assuming that in the studied temperature range they are constant. Calculate the "melting" temperature T_m of this oligonucleotide, i.e. the temperature at which exactly half of the dimer is dissociated. Take the following values of molar absorption coefficients: $\epsilon_M = 8.08 \cdot 10^4\,\text{cm}^{-1}\text{M}^{-1}$ for the oligonucleotide in monomer form and $\epsilon_D = 1.02 \cdot 10^5\,\text{cm}^{-1}\text{M}^{-1}$ for the oligonucleotide in dimer form. The gas constant is $R = 8.314\,\text{J}/(\text{mol} \cdot \text{K})$. It is useful to remember that $1\,\text{M} = 1\,\text{mol}/\text{dm}^3$.

Problem 21

A coherent state of a one-dimensional quantum harmonic oscillator with frequency ω, denoted by $|\alpha\rangle$, is defined as an eigenstate of the annihilation operator $\hat{a}$, i.e. $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, where α is a complex number and

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{a_{\text{ho}}} + i\frac{a_{\text{ho}}}{\hbar}\hat{p}\right) \quad \text{and} \quad a_{\text{ho}} = \sqrt{\frac{\hbar}{m\omega}}\,.$$

Determine the position representation of this state (omitting detailed calculations of the normalization constant) and show that no eigenstate of the creation operator $\hat{a}^\dagger$ exists.