Doctoral School of Exact and Natural Sciences
Physical Sciences – written exam

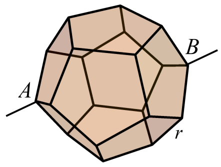
Instruction manual. Solving the problems, present the reasoning behind the result. Give your final numerical results with the accuracy of two or three digits, after rounding up appropriately, e.g.. $1.23456 \cdot 10^{-19} \approx 1.23 \cdot 10^{-19}$.
Problems 1-8 are the easier problems. Turn in solutions of only **four** of these problems. For each solution, you can be assigned up to 6 points.
Problems 9-14 are the harder problems. Turn in solutions of only **two** of these problems. For each solution, you can be assigned up to 8 points.

Prepare solutions for different problems on separate sheets.

EASIER PROBLEMS

Problem 1

The edges of a regular dodecahedron are made of resistive wire. The resistance of each edge is equal to r. A current source is connected to two opposite vertices A and B. Determine the equivalent resistance R_{AB} of this dodecahedron.



Problem 2

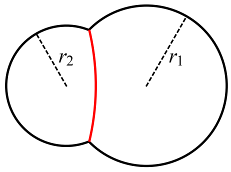
A horizontal, weightless rod of length $2a$ can rotate freely around a vertical axis passing through its center. Two identical small beads, each of mass m, are threaded onto the rod. They can move along the rod without friction and rebound elastically from stoppers placed at the ends of the rod. Initially, the beads are fixed at distances $\frac{1}{2}a$ from the axis of rotation. The entire system is then spun up to an angular velocity ω_0, after which the beads are released. **(a)** What path will the beads follow? **(b)** After what time will the rod complete a full 360° rotation?

Problem 3

A hemispherical dome rests snugly on the surface of a table. Water is poured into the dome through a hole at its top. When the water level reaches the hole, the dome is lifted and water begins to leak out from underneath. Determine the mass M of the dome if its radius is $R = 30\,\text{cm}$.

Problem 4

Two connected soap bubbles have the shapes of spherical caps with radii $r_1 = 4\,\text{cm}$ and $r_2 = 3\,\text{cm}$, respectively. The film separating the two bubbles is also a part of a sphere. What is the radius r_{12} of this spherical surface? The surface tension of the soap water from which the bubbles are made is $\sigma = 0.03\,\text{N/m}$.



Problem 5

High-frequency sound waves (e.g., ultrasound) can propagate in media as narrow beams. Consider an ultrasound beam traveling through air that strikes the surface of water at an angle $\alpha = \frac{\pi}{4}$. Determine what fraction of the beam's intensity is reflected from the water surface. Assume that the speed of sound in air is $v_\text{a} = 340\,\text{m/s}$, and the speed of sound in water is $v_\text{w} = 1450\,\text{m/s}$.

Problem 6

The quantum states $|A\rangle$ and $|B\rangle$ of a certain physical system are eigenstates of the Hamiltonian corresponding to energies $E_A = E$ and $E_B = 2E$, respectively. At time $t = 0$, the system is prepared in the state:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|A\rangle - \frac{1}{\sqrt{2}}|B\rangle.$$

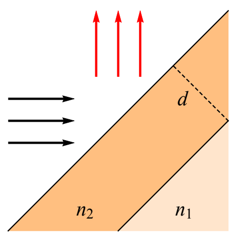
Determine the minimal time $t > 0$ for which the time-evolved state $|\psi(t)\rangle$ is orthogonal to the initial state $|\psi(0)\rangle$.

Problem 7

The frequency ratio between successive semitones in the musical scale is $2^{1/12} \approx 1.059$. Assume that a sound is perceived as pure if it deviates from its intended frequency by no more than 1/8 of a semitone. How long must a tone of frequency $\nu_1 = 880\,\text{Hz}$ sung by a soprano, and a tone of frequency $\nu_2 = 110\,\text{Hz}$ sung by a bass, last at minimum so that the listener can perceive the pitch accuracy of the aria performance?

Problem 8

A beam of white light is incident at an angle of 45° onto a glass plate with refractive index $n_1 = 1.5$, which is coated with a thin film of refractive index $n_2 = 1.8$. What should be the thickness d of the thin film in order to obtain a maximum reflection coefficient for red light and a minimal reflection coefficient for light in the blue-to-green range?

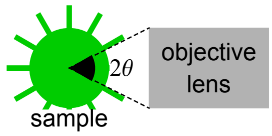


— t u r n t h e p a g e —

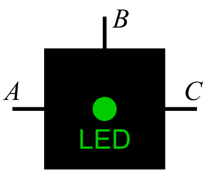
HARDER PROBLEMS

Problem 9
Proton radioactivity is a decay process in which a nucleus emits a proton. **(a)** Calculate the kinetic energy of the proton and the energy of the recoiling nucleus produced in the proton decay of ${}^{151}_{71}\mathrm{Lu}$ to the ground state of ${}^{150}_{70}\mathrm{Yb}$. **(b)** What orbital angular momentum is carried away by the emitted proton? The mass excess of ${}^{151}_{71}\mathrm{Lu}$ is $-30.108\,\mathrm{MeV}$, the mass excess of ${}^{150}_{70}\mathrm{Yb}$ is $-38.830\,\mathrm{MeV}$, and the mass excess of ${}^{1}\mathrm{H}$ is $7.289\,\mathrm{MeV}$. The spin and parity of ${}^{151}_{71}\mathrm{Lu}$ are $\frac{11}{2}^-$, while the ground state of the even-even daughter nucleus has spin and parity 0^+.

Problem 10
In order to measure the polarization of the electron beam at the LEP accelerator at CERN, Compton backscattering was performed using a laser beam. A beam of photons from a Nd:YAG laser with a wavelength of $\lambda = 532\,\mathrm{nm}$ was scattered off a beam of electrons with energy $E = 45\,\mathrm{GeV}$. Calculate the energy of photons that are backscattered (i.e., scattered at an angle of 180°). Assume $\hbar c = 197\,\mathrm{MeV}\cdot\mathrm{fm}$.

Problem 11


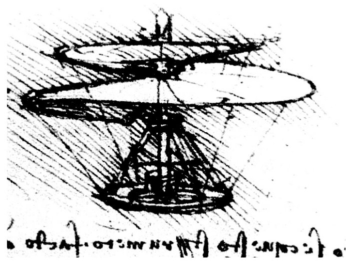
Laser light with a wavelength of 485 nm is used to excite fluorescence at a wavelength of 497 nm in a sample located in the focal plane of an oil-immersion objective lens with a numerical aperture of 1.4. The fluorescence is detected by a single-photon detector. If too many photons reach the detector in too short a time, the detector will shut down. Determine the maximum laser power (measured just before the objective) that will not lead to detector shutdown in the above experimental setup. Assume the following: the detector can safely register up to 10^7 photons per second; the probability that a fluorescence photon is detected in the detector plane is 0.25; the probability that a fluorophore molecule absorbs an incoming photon is 10^{-6}; the probability of photon emission following absorption is 0.5; the absorption probability is constant and independent of laser power; the sample emits fluorescence isotropically; the refractive index of the immersion oil is $n = 1.518$. For optical instruments, the numerical aperture NA is defined by the formula $NA = n\sin\theta$, where θ is half the maximum angle under which light can enter the instrument, and n is the refractive index of the medium in which the instrument is immersed.

Problem 12


The following elements are provided: a black box with a visible green LED and three electrical terminals labeled A, B, and C; a broken multimeter, whose ohmmeter function is inoperative, but whose voltmeter and ammeter functions work correctly; a battery with a known electromotive force; and an arbitrary number of wires. In addition to the LED, the black box contains two resistors, one of which has three times the resistance of the other. When the battery terminals are connected to each pair of contacts, the following effects are observed, as shown in the table below.

Positive +	Negative −	LED state	Positive +	Negative −	LED state
A	B	does not light	A	C	does not light
B	A	lights very brightly	B	C	lights dimly
C	A	does not light	C	B	does not light

Propose a method to determine the arrangement of the components inside the box, given that no resistors are connected in parallel to any of the terminals, and that none of the terminals is electrically isolated from the rest of the circuit. The box must not be opened, damaged, or destroyed in any way.

Problem 13


Among Leonardo da Vinci's drawings is a design for a helicopter powered by human muscular effort. Assume that the rotor diameter of such a helicopter is $d = 8\,\mathrm{m}$, and the total mass of the helicopter with the pilot is $m = 80\,\mathrm{kg}$. Could this helicopter lift off the ground if, under sustained exertion, the pilot can generate a power output of $P = 150\,\mathrm{W}$? Assume that the air density is $\rho = 1.3\,\mathrm{kg/m^3}$.

Problem 14
A rod of mass M and length L is placed in a constant and uniform gravitational field of intensity g. The rod is suspended at both ends on two vertical springs with identical free lengths and equal spring constants k. The springs remain vertical at all times, so the motion of the rod takes place in a plane. The rod performs small oscillations around its equilibrium position. Determine the frequencies and normal modes of t