WARSAW DOCTORAL SCHOOL OF MATHEMATICS AND COMPUTER SCIENCE

July 2, 2024

Entrance exam

On the following pages you will find 16 problems related to various areas of Mathematics and Computer Science. You are expected to choose and solve **any 4 of them**. Each problem is worth the same number of points.

You are free to choose any problems you wish, i.e. candidates for studies in Mathematics may also choose Computer Science problems, and *vice versa*.

Most of the problems are composed of a few subproblems, but each problem, i.e. all of its subproblems, are graded as a whole.

You may attempt to solve more than 4 problems. All your solutions will be graded, but **only 4 best-graded solutions will contribute to your general grade**.

All your answers should be appropriately justified. **Every problem should be solved on a separate sheet of paper**; of course, the solution of one problem can be written on more that one sheet.

Each sheet of paper should be **signed with your first name and surname**, and **marked with the problem number**.

Exam duration: 3 hours

Good luck!

Analysis

PROBLEM 1. We define a functional sequence $f_n\colon [0,+\infty) \to \mathbb{R}$ by a formula $f_n(x) = \dfrac{1}{\sqrt[n]{1+x^n}}$.

(a) Find a limit of the sequence $(f_n)_{n\in\mathbb{N}}$. Determine if the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ is uniform on $[0,\infty)$. Is the convergence uniform on $[0,1]$?

(b) Does the sequence $(f_n')_{n\in\mathbb{N}}$ (of derivatives of f_n) converge uniformly on $[0,2024]$?

(c) Let $A = \{(x,y) \in \mathbb{R}^2 : 0 < x < 2, 0 < y < 2\}$. Calculate $\lim_{n\to\infty} \int_A f\left(\frac{2x}{y}\right) x \mathrm{d}\lambda_2(x,y)$, where λ_2 is the two dimensional Lebesgue measure.

(d) Justify that for some neighbourhood of $(x_0,y_0) = (1,1)$ an equation

$$\ln\sqrt{x^2+y^2} - \operatorname{arc\,tg}\frac{x}{y} = \frac{1}{2}\log 2 - \frac{\pi}{4}$$

uniquely determines function $y(x)$ defined on an interval $(1-\varepsilon, 1+\varepsilon)$ for some $\varepsilon > 0$, which is C^2 class. Calculate $y'(1)$. Prove that y has a local extreme at $x = 1$ and determine if the extreme is minimum or maximum.

Complex analysis

PROBLEM 2. Fix $r > 0$ and complex numbers $a, b \in \mathbb{C}$, $a \neq b$ with $|a|, |b| < r$. Define a Möbius transformation h by the formula

$$h(z) = \frac{z-a}{z-b}.$$

(a) Show that for some $t \in (0,r)$, there exists a holomorphic map $f\colon \{z \in \mathbb{C}\colon |z| > t\} \to \mathbb{C}$ satisfying $\exp(f(z)) = h(z)$ for every $z \in \mathbb{C}$, $|z| > t$.

(b) Assume now that a, b are real numbers and $0 < b < a < r$. Define $C(0,r) = \{z \in \mathbb{C}\colon |z| = r\}$. Show that the image of the circle $C(0,r)$ under the homography h is a circle contained in the half-plane $\{z \in \mathbb{C}\colon \operatorname{Re} z > 0\}$.

(c) Denote by $\operatorname{Log}\colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ the principal branch of logarithm defined as $\operatorname{Log}(z) = \log|z| + \mathrm{i}\operatorname{Arg}(z)$, where $\operatorname{Arg}(z) \in (-\pi,\pi]$ is the principal argument of a complex number z. Define a function

$$f(z) = \exp\left(\frac{1}{z}\right) \cdot \operatorname{Log}\frac{z-2}{z-1}.$$

Calculate the residue $\operatorname{Res}(f,\infty)$ of f at infinity and the integral

$$\int\limits_{C(0,r)} f(z)\,\mathrm{d}z,$$

where the circle $C(0,r)$ is oriented counterclockwise.

Probability and statistics

PROBLEM 3. Let X, Y be independent and identically distributed random variables. By $\sigma(X+Y)$ we denote the σ-algebra generated by the random variable $X+Y$; the symbol $\mathbb{1}_A$ stands for the characteristic function of the set A.

(a) Show that for every $A \in \sigma(X+Y)$, we have $\mathbb{E}(\mathbb{1}_A X) = \mathbb{E}(\mathbb{1}_A Y)$.

(b) Determine all pairs (X, Y) for which the equality $\mathbb{E}(X^2 \mid X+Y) = XY$ holds almost everywhere.

(c) Assume additionally that X and Y have the exponential distribution with parameter $\lambda > 0$, i.e. the distribution with the density function $g(x) = \lambda e^{-\lambda x} \mathbb{1}_{(0,\infty)}(x)$. For any fixed $y \in (0, \infty)$, calculate the conditional expectation

$$\mathbb{E}(|X - Y| \mid Y = y).$$

(d) Assume that $(X_n)_{n=1}^{\infty}$ is a sequence of independent random variables with the same exponential distribution. For $n \in \mathbb{N}$, define a random variable Y_n by

$$Y_n = n^{\frac{2025}{2024}} \cdot \min\{X_1, \dots, X_n\}.$$

Decide whether $(Y_n)_{n=1}^{\infty}$ is convergent in distribution, that is, whether there exists a random variable U such that $Y_n \xrightarrow{D} U$.

Geometry and linear algebra

Problem 4. Let V be an n-dimensional linear space over the field of complex numbers. Fix an endomorphism f of V. Define the operator $\Phi_f \colon \mathrm{End}_{\mathbb{C}}(V) \to \mathrm{End}_{\mathbb{C}}(V)$ by the following formula

$$\Phi_f(h) = f \circ h.$$

(a) Find the Jordan normal form of Φ_f. Note its dependence on the Jordan normal form of the endomorphism f.

(b) Suppose f has a square root, that is suppose there exists endomorphism g such that $g^2 = f$. Classify endomorphism that have a square root. Provide an example of an endomorphism that has no square root.

(c) Verify that, in the setting of the part (b), Φ_g is the square root of Φ_f. Provide an example of an operator Φ_f and its square root, which is not of the form Φ_g for any $g \in \mathrm{End}_{\mathbb{C}}(V)$.

(d) We denote by f^* the Hermitian conjugate of f with some Hermitian product on V. Check that the formula $q(f, g) = \mathrm{tr}(f^* \cdot g)$ defines Hermitian product on $\mathrm{End}_{\mathbb{C}}(V)$. Classify $f \in \mathrm{End}_{\mathbb{C}}$ such that the operator Φ_f is Hermitian with respect to q?

Algebra

Problem 5. Let us recall $SL(2, \mathbb{C})$ is the group of 2×2 matrices with complex coefficients and determinant 1 with multiplication of matrices as the group law.

We pick the following three matrices

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

(a) Identify the subgroup $H \leqslant SL(2, \mathbb{C})$ generated by I, J, K, that is, find the order of the group, the relations between the generators and find all its subgroups and homomorphic images.

(b) Let G be a group whose all subgroups are normal. Classify such groups of order 72.

(c) Is every subgroup of $H \times \mathbb{Z}_4$ normal?

(d) Let $\mathbb{R}[x, y]$ be the ring of polynomials in two variables and $\mathcal{C}(S^1, \mathbb{R})$ the ring of continuous functions with real values on the unit circle which is the zero set in $\mathbb{R}^2$ of the equation $x^2+y^2 = 1$. Describe the kernel of the homomorphism $\varphi \colon \mathbb{R}[x, y] \to \mathcal{C}(S^1)$ defined to be $\varphi(f) = f_{|S^1}$ sending f to its restriction to S^1. Is φ epimophic? Is $\mathbb{R}[x, y]$ is a domain? Is $\mathcal{C}(S^1, \mathbb{R})$ a domain?

Topology

PROBLEM 6. Let I stand for the interval $[0,1]$ equipped with the Euclidean topology, and let $\mathbf{C} \subset [0,1]$ be the Cantor set. Let also λ be the one-dimensional Lebesgue measure.

(a) Decide whether there exists an infinite family $\{P_\alpha \colon \alpha \in A\}$ of pairwise disjoint closed intervals with nonempty interiors contained in I such that

$$\lambda\Big(I \setminus \bigcup_{\alpha \in A} P_\alpha\Big) = 0.$$

(b) Show that there is no infinite family $\{P_\alpha \colon \alpha \in A\}$ of pairwise disjoint closed intervals with nonempty interiors such that $\bigcup_{\alpha \in A} P_\alpha = I$.

(c) Let $F \subset I$ be a closed set with empty interior. Prove that there exists a homeomorphism $\varphi \colon I \to I$ such that $\lambda(\varphi(F)) = 0$.

(d) It is known that there is a continuous surjection $f \colon \mathbf{C} \to I$. Using the function f construct a homeomorphism $\varphi \colon I \to I$ satisfying $\lambda(\varphi(\mathbf{C})) = \frac{1}{2}$.

Ordinary differential equations

PROBLEM 7. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a continuous function. We consider a system of ordinary differential equations

$$(\spadesuit) \qquad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t) - x, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = (x-1)y^\alpha, \end{cases}$$

for some constant $\alpha > 0$.

(a) Assume that $f(t) < 0$ for all $t \in \mathbb{R}$. For which values of paramters $\alpha > 0$, $x_0 \geqslant 0$, $y_0 \geqslant 0$ are solutions of system ($\spadesuit$) with an initial condition $x(0) = x_0$, $y(0) = y_0$ unique?

(b) Let $f(t) = 2$ for all $t \in \mathbb{R}$. We consider solutions of system ($\spadesuit$) with an initial condition $x(0) = 2$, $y(0) = 4$. For which values of the parameter $\alpha \in \mathbb{R}$, $\alpha > 0$ are these solutions defined for all $t \in (0, +\infty)$?

(c) For the constant function $f(t) \equiv a \in \mathbb{R}$ and $\alpha = 1$, find all stationary states of system ($\spadesuit$) and examine their local stability.

(d) Assume that f periodic function with basic period T (so T is the smallest positive real number such that $f(t) = f(t+T)$ for all $t \in \mathbb{R}$).
   (i) Prove there exists exactly one $x_0 \in \mathbb{R}$, such that solutions of the first equation of system ($\spadesuit$) with the initial condition x_0 is a periodic function with a period T.
   (ii) Let x^* be a solution described in the previous point (i). Prove that mean value of the function x^* on an arbitrary interval $[t, t+T]$ equals to the mean value of f on the interval $[0,T]$. This is

$$\frac{1}{T}\int_t^{t+T} x^*(s)\mathrm{d}s = \frac{1}{T}\int_0^T f(s)\mathrm{d}s.$$

Functional analysis

PROBLEM 8. Let S be the real vector space consisting of all sequences $(x_n)_{n=1}^{\infty}$ for which the series $\sum_{n=1}^{\infty} x_n$ converges, and equipped with the standard, coordinatewise operations of addition and scalar multiplication. By c_0 we denote the classical Banach space of sequences converging to zero, with the supremum norm. Consider a norm $\|\cdot\|_{\mathsf{S}}$ on S defined by the formula

$$\|\mathbf{x}\|_{\mathsf{S}} = \sup_{N\in\mathbb{N}} \left| \sum_{k=1}^{N} x_k \right| \qquad \text{for } \mathbf{x} = (x_n)_{n=1}^{\infty} \in \mathsf{S}.$$

(a) Show that $(\mathsf{S}, \|\cdot\|_{\mathsf{S}})$ is a Banach space.

(b) Find a linear isomorphism (i.e. an invertible bounded linear operator) between the spaces S and c_0. Justify your answer, that is, prove that a given operator is indeed a linear isomorphism.

(c) Show that for every functional $\varphi \in \mathsf{S}^*$ there is a sequence $(y_n)_{n=1}^{\infty} \in \ell_1$ such that

$$\varphi(\mathbf{x}) = \sum_{n=1}^{\infty} y_n \Big(\sum_{k=n}^{\infty} x_k \Big) \quad \text{for } \mathbf{x} = (x_n)_{n=1}^{\infty} \in \mathsf{S}.$$

(d) Assume that a sequence of functionals $(x_n^*)_{n=1}^{\infty} \subset \mathsf{S}^*$ has the property $\sum_{n=1}^{\infty} |x_n^*(\mathbf{x})| < \infty$ for each $\mathbf{x} \in \mathsf{S}$. Prove that there exists a constant $C > 0$ such that

$$\sum_{n=1}^{\infty} |x_n^*(\mathbf{x})| \leqslant C\|\mathbf{x}\|_{\mathsf{S}} \quad \text{for every } \mathbf{x} \in \mathsf{S}.$$

Programming languages

PROBLEM 9. The problem consists of four independent tasks:

(1) Consider the following function in C:

```
int* fib(int n) {
    int tab[n];

    if (n <= 0) {
        return NULL;
    }

    tab[0] = 0;
    if (n > 1) {
        tab[1] = 1;
    }

    for (int i = 2; i < n; i++) {
        tab[i] = tab[i-1] + tab[i-2];
    }
    return tab;
}
```

What problems do you see with this function? How to fix them?

(2) Consider the following code in C++:

```
struct TreeNode {
    int val;
    TreeNode* left;
    TreeNode* right;

    TreeNode(int x) : val(x), left(nullptr), right(nullptr) {}
};

void inorderTraversal(TreeNode* root) {
    inorderTraversal(root->left);
    std::cout << root->val << " ";
    inorderTraversal(root->right);
}

int main() {
    TreeNode* root = new TreeNode(1);
    root->left = new TreeNode(2);
    root->right = new TreeNode(3);
    root->left->left = new TreeNode(4);
    root->left->right = new TreeNode(5);

    inorderTraversal(root);

    return 0;
}
```

What problems do you see with the above code? How to fix them? Write equivalent code without using recursion.

(3) Consider the following code in OCaml:

```
let f g a b s =
  let rec h x acc =
    if x >= b then
      acc
    else
      let py = g x in
      let y = g (s x) in
      let ny = g (s (s x)) in
      if py < y && ny < y then
        h (s x) ((s x) :: acc)
      else
        h (s x) acc
  in
  h a []
```

What does the function `f` do? What is the type of this function?

(4) Write a Python decorator that counts the number of calls to a given function. Write a program fragment that uses such a decorator.

Discrete mathematics

PROBLEM 10. *Remark.* A closed-form solution is expected in all subproblems of this problem.

(a) How many ways are there to divide a $2 \times n$ strip into 2×1 rectangles? (Here, and in the remainder of this problem, pieces can only overlap at the edges.)

(b) How many ways are there to divide a $2 \times n$ strip into 2×1 and 2×2 rectangles?

(c) Let $a_{n,k}$ be the number of ways to divide a $2 \times n$ strip into 2×1 and 2×2 rectangles, in which exactly k pieces are 2×2. Let $A(x) = \sum_{n=2k}^{\infty} a_{n,k} x^n$. Provide a closed form for $A(x)$.

(d) Let b_n be the number of ways to divide a $2 \times n$ strip into 2×1 and 2×2 rectangles and the following shapes $L = [0, 2] \times [0, 2] \setminus [0, 1) \times [0, 1)$? Let $B(x) = \sum_{n=0}^{\infty} b_n x^n$. Provide a closed form for $B(x)$.

Algorithms and data structures

PROBLEM 11. For $M \in \mathbb{N}$ and a family $S_1, \ldots, S_n \subseteq \{0, \ldots, M\}$ of n sets, a *hitting interval* for these sets is an integral interval $I = \{a, a + 1, \ldots, b\}$, $0 \leqslant a \leqslant b \leqslant M$ such, that $I \cap S_i \neq \emptyset$ for all i.

(a) Give an $O(n)$ algorithm for finding a shortest hitting interval when each S_i is an interval $S_i = \{a_i, \ldots, b_i\}$, given by its endpoints a_i, b_i.

(b) Give an $O(n)$ algorithm deciding, for a given k and family of 2-element sets $\{S_i\}_{i=1,\ldots,n}$, whether there exists a hitting interval of length k. Here, we assume $M = O(n)$.

(c) Give an $O(n)$ algorithm which, for a given family of 2-element sets $\{S_i\}_{i=1,\ldots,n}$, finds for every $k \in \{0, \ldots, M\}$ the number of different length k hitting intervals. Again, we assume $M = O(n)$.

(d) Give an $O(n)$ algorithm which, for a given family of 2-element sets $\{S_i\}_{i=1,\ldots,n}$, finds the total number of different hitting intervals. Here, we assume $M = O(n^2)$.

Remark. You need to justify the correctness of your algorithms and estimate their time complexity.

Logic and databases

PROBLEM 12. Give a formula ϕ of first-order logic over some signature Σ with the following property, or show that no such formula exists. In each subtask we count models up to isomorphism (two models are isomorphic if and only if there exists a bijection between their universes that keeps the interpretation of every symbol in Σ).

(a) a formula having exactly $n/2$ models of cardinality n, for each even $n > 0$, and having no models of cardinality n for odd n;

(b) a formula having exactly one infinite model;

(c) a formula having exactly one finite model and infinitely many infinite models;

(d) a formula ϕ over the signature consisting of a single functional symbol f of arity 1, such that for every finite $n > 0$, the only model of ϕ is the structure with the universe $\{0, \ldots, n-1\}$ where f is interpreted as the function $i \mapsto (i + 1) \bmod n$.

Automata and formal languages

PROBLEM 13. Consider words over the alphabet $\Sigma = \{1, 2\}$. Let S_n be the language of words $w \in \Sigma^*$ of sum n (e.g., $S_4 = \{22, 211, 121, 112, 1111\}$). For each of the following languages, answer whether they are regular, and whether they are context-free. For the regular languages, give the order of the smallest number of states of a finite automaton recognizing this language, depending on n (e.g., $\Theta(n^2)$ states), both for deterministic and nondeterministic finite automata.

(a) $S_n\Sigma^*$
(b) $\Sigma^* S_n$
(c) $\bigcup_{k\geqslant n} S_k S_k$
(d) $\bigcup_{k\geqslant n} S_k S_k S_k$

Computation theory and computational complexity

Problem 14. The 3-cycle graph covering problem is defined as follows. For a directed graph $G = (V, E)$ and an integer L, we ask whether there exists a set of triples of vertices S, $|S| \geqslant L$, such that each triple $t \in S$ is a cycle in G and each vertex from V belongs to at most one triple in S.

(a) Prove that the 3-cycle graph covering problem is NP-complete.

(b) Prove that there exists a polynomial 3-approximation algorithm for the 3-cycle graph covering problem.

(c) Provide a polynomial approximation algorithm with an approximation ratio better than 3 for the case where our goal is to cover the graph with cycles of size at most 3 and each edge in the graph is bidirectional (if there is an edge from u to v, then there is also an edge from v to u).

(d) Prove that the problem is FPT (fixed-parameter-tractable) for connected graphs with respect to the parameter $(|E| - |V|)$.

Hint. Consider the following three-dimensional matching problem. Given sets X, Y, and Z, a set T which is a subset of $X \times Y \times Z$ (in other words, T consists of triples (x, y, z) such that $x \in X$, $y \in Y$, and $z \in Z$), and an integer L. We say that $M \subseteq T$ is a three-dimensional matching if the following conditions hold: for each pair of different triples $(x_1, y_1, z_1) \in M$ and $(x_2, y_2, z_2) \in M$, we have $x_1 \neq x_2$, $y_1 \neq y_2$, and $z_1 \neq z_2$. We ask whether there exists a three-dimensional matching of size at least L. This problem is NP-hard.

Concurrent and distributed programming, computer systems

Problem 15. A distributed system conducts a numerical simulation of a physical phenomenon that uses a ring geometry. Processes communicate with the following interface: `send(recipient, message)` and `message = receive([sender])` (when `sender` is not set, a message from any sender is received). The communication is synchronous, using the rendez-vous semantics: `send` blocks until the message is received by the receiver; and `receive` blocks until the message is sent by the sender. Communication and processes are failure-free. On each pair of communicating processes, the messages are received in the same order as they were sent. If many senders send messages to the same receiver, `receive` guarantees weak fairness.

Solve the following problems. For each solution, specify the class (the type) of the problem.

(1) Assume that the system correctly initializes the local state of the simulation and the variables describing the predecessor (`pid_prev`) and the successor (`pid_next`) of each process so that the processes together form a ring. There are local functions `summarize` (summarizing the state to a shorter information); and `evolve` (evolving into an updated state the current local state and the information from the predecessor and the successor). Prove that the following code is correct, or show a counterexample and fix the code (you do not have to formally prove your version is correct).

```
process compute(int pid, int pid_prev, int pid_next) {
  State s = <initial_local_state>;
  for (int step = 0; step < K; step++) {
    Value x = summarize(s);
    send(pid_prev, x);
```

```
    Value x_prev = receive()
    send(pid_next, x);
    Value x_next = receive();
    s = evolve(s, x_prev, x_next);
  }
```

(2) Now assume that the system has many potential workers that execute the following code (assume that `compute()` is a local function with an identical code to the process from the previous sub-problem). Write the code of the `Coordinator` process that joins P ($P > 2$) processes into a ring.

```
    process worker(int pid) {
    while (true) {
      rest();
      send(Coordinator, pid);
      pid_prev = receive();
      pid_next = receive();
      compute(pid, pid_prev, pid_next);
    }
  }
```

(3) Now assume that during the computation, occasionally a process wants to save a snapshot of its local state on a *shared* output device. The snapshot should be created immediately after calling `evolve()`, when a local function `bool takeSnapshot()` returns true. A process saves a snapshot by calling a local function `write(byte[])`. As the state is large, the function must be called many times: `for (part in s.parts()) write(part.asBytes())`. Extend the code of `compute` function so that the snapshots on the output device written by different processes are not interweaved.

Bioinformatics

Problem 16.

(a) Given the distance matrix below, construct a phylogenetic tree for species A-E using the UPGMA algorithm. Label all tree branch lengths.

	A	B	C	D	E
A	0	15	30	20	100
B		0	25	35	5
C			0	10	40
D				0	125
E					0

(b) Determine at least one optimal global alignment for the sequences CTTAAG and CTAAT using the following scoring scheme: +2 for a match, -1 for a mismatch, and -2 for a gap and the Needleman-Wunsch dynamic programming algorithm. Write down the alignment, the alignment score and the dynamic programming matrix.

(c) How would you evaluate the results of data clustering performed with the k-means algorithm? Propose two methods and briefly discuss them.